INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 5.6-5.7]

RULES OF DIFFERENTIATION

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A. Paper 1 questions (SHORT)

BASIC RULES OF DIFFERENTIATION

1. [Maximum mark: 3 per function]

Differentiate the following functions:

$y=7x^3-2-5e^x-3\sin x$	
$y=\sqrt{x}+\ln x$	
$y=\sqrt{x}\ln x$	
$y=\dfrac{\ln x}{\sqrt{x}}$	
$y=\dfrac{2x+1}{3x-5}$	
$y=x+ex+\ln\pi$	
$y=x^2+\ln x+x^2\ln x$	
$y=x\sin x\ln x$	
$y=x^2e^x\ln x$	

2. [Maximum mark: 4]

Let $f(x) = 2x^3 + \ln x$

(a) Find $f'(x)$. [2]

(b) Find the gradient of the curve $y = f(x)$ at $x = 1$. [2]

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3. [Maximum mark: 6]

Let $f(x) = \dfrac{x^3 + 1}{\sin x}$

(a) Find $f'(x)$. [3]

(b) Find the gradient of the curve $y = f(x)$

(i) at $x = \dfrac{\pi}{4}$ (ii) at $x = 1$ rad. [3]

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4. [Maximum mark: 12]

Given the following values at $x = 1$

x	$f(x)$	$g(x)$	$f'(x)$	$g'(x)$
1	2	3	4	5

Calculate the derivatives of the following functions at $x = 1$

(i) $y = 3f(x) - 2g(x)$

(ii) $y = f(x)g(x)$

(iii) $y = \frac{f(x)}{g(x)}$

(iv) $y = 2x^3 + 1 + 5f(x)$

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5. [Maximum mark: 4]

Let $f(x) = 6\sqrt[3]{x^2}$. Find $f'(x)$.

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6. [Maximum mark: 6]

Let $h(x) = \frac{6x}{\cos x}$. Find $h'(0)$

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7. [Maximum mark: 5]

Let $g(x) = 2x \sin x$.

(a) Find $g'(x)$. [3]

(b) Find the **exact** value of the gradient of the graph of g at $x = \pi$. [2]

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8. [Maximum mark: 4]

Consider the function $f(x) = k\sin x + 3x$, where k is a constant.

(a) Find $f'(x)$. [2]

(b) When $x = \frac{\pi}{3}$, the gradient of the curve of $f(x)$ Is 8. Find the value of k. [2]

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9. [Maximum mark: 5]

Let $f(x) = \frac{3x^2}{5x-1}$.

(a) Write down the **equation** of the vertical asymptote of $y = f(x)$.

(b) Find $f'(x)$. Give your answer in the form $\frac{ax^2+bx}{(5x-1)^2}$ where a and $b \in \mathbb{Z}$.

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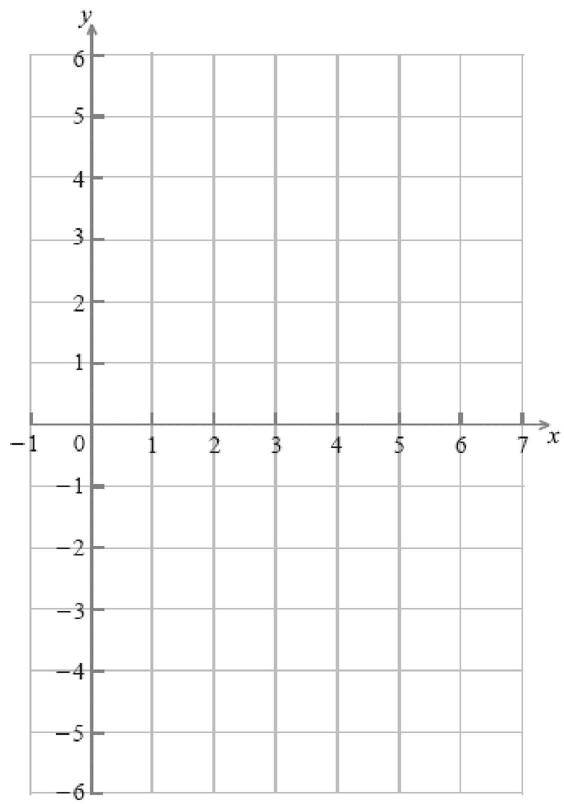
10. [Maximum mark: 8]

Let $f(x) = x\cos x$, for $0 \leq x \leq 6$.

(a) Find $f'(x)$. [3]

(b) On the grid below, sketch the graph of $y = f'(x)$. [3]

(c) Write down the range of the function $y = f'(x)$, for $0 \leq x \leq 6$ [2]



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11. [Maximum mark: 7]

Let $f(x) = e^x \cos x$.

(a) Find $f'(x)$. [3]

(b) Find the gradient of the normal to the curve of f at $x = \pi$. [2]

(c) Find the gradient of the tangent to the curve of f at $x = \frac{\pi}{4}$. [2]

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12. [Maximum mark: 7]

Let $f(x) = xe^x$.

(a) Find the equation of the tangent line at $x = 0$. [3]

(b) Find the equation of the normal line at $x = 0$. [2]

(c) Solve the equation $f'(x) = 0$. [2]

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13. [Maximum mark: 3 per function]

Find the derivative of each function below. *[cover 3rd column; then compare with your answer]*

Function $f(x)$	Derivative $f'(x)$	Compare with correct answer
$f(x)=e^{x-5}$		$f'(x)=e^{x-5}$
$f(x)=e^{2x}$		$f'(x)=2e^{2x}$
$f(x)=e^{2x-5}$		$f'(x)=2e^{2x-5}$
$f(x)=e^{5-2x}$		$f'(x)=-2e^{5-2x}$
$f(x)=e^{2x^2-5}$		$f'(x)=4xe^{2x^2-5}$
$f(x)=e^{2x^7-5}$		$f'(x)=14x^6e^{2x^7-5}$
$f(x)=\sin(x-5)$		$f'(x)=\cos(x-5)$
$f(x)=\sin 2x$		$f'(x)=2\cos 2x$
$f(x)=\sin(2x-5)$		$f'(x)=2\cos(2x-5)$
$f(x)=\sin(5-2x)$		$f'(x)=-2\cos(5-2x)$
$f(x)=\sin(2x^2-5)$		$f'(x)=4x\cos(2x^2-5)$
$f(x)=\sin(2x^7-5)$		$f'(x)=14x^6\cos(2x^7-5)$
$f(x)=\cos(x-5)$		$f'(x)=-\sin(x-5)$
$f(x)=\cos 2x$		$f'(x)=-2\sin 2x$
$f(x)=\cos(2x-5)$		$f'(x)=-2\sin(2x-5)$
$f(x)=\cos(5-2x)$		$f'(x)=2\sin(5-2x)$
$f(x)=\cos(2x^2-5)$		$f'(x)=-4x\sin(2x^2-5)$
$f(x)=\cos(2x^7-5)$		$f'(x)=-14x^6\sin(2x^7-5)$

Function $f(x)$	**Derivative** $f'(x)$	**Compare with correct answer**
$f(x)=\tan(x-5)$		$f'(x)=\dfrac{1}{\cos^2(x-5)}$
$f(x)=\tan 2x$		$f'(x)=\dfrac{2}{\cos^2(2x)}$
$f(x)=\tan(2x-5)$		$f'(x)=\dfrac{2}{\cos^2(2x-5)}$
$f(x)=\tan(5-2x)$		$f'(x)=\dfrac{-2}{\cos^2(5-2x)}$
$f(x)=\tan(2x^2-5)$		$f'(x)=\dfrac{4x}{\cos^2(2x^2-5)}$
$f(x)=\tan(2x^7-5)$		$f'(x)=\dfrac{14x^6}{\cos^2(2x^7-5)}$
$f(x)=\ln(x-5)$		$f'(x)=\dfrac{1}{x-5}$
$f(x)=\ln 2x$		$f'(x)=\dfrac{2}{2x}=\dfrac{1}{x}$
$f(x)=\ln(2x-5)$		$f'(x)=\dfrac{2}{2x-5}$
$f(x)=\ln(5-2x)$		$f'(x)=\dfrac{-2}{5-2x}=\dfrac{2}{2x-5}$
$f(x)=\ln(2x^2-5)$		$f'(x)=\dfrac{4x}{2x^2-5}$
$f(x)=\ln(2x^7-5)$		$f'(x)=\dfrac{14x^6}{2x^7-5}$
$f(x)=\sqrt{x-5}$		$f'(x)=\dfrac{1}{2\sqrt{x-5}}$
$f(x)=\sqrt{2x-5}$		$f'(x)=\dfrac{1}{\sqrt{2x-5}}$
$f(x)=\sqrt{5-2x}$		$f'(x)=\dfrac{-1}{\sqrt{5-2x}}$
$f(x)=\sqrt{2x^2-5}$		$f'(x)=\dfrac{2x}{\sqrt{2x^2-5}}$
$f(x)=\sqrt{2x^7-5}$		$f'(x)=\dfrac{7x^6}{\sqrt{2x^7-5}}$

Function $f(x)$	**Derivative** $f'(x)$	**Compare with correct answer**
$f(x)=(x-5)^3$		$f'(x)=3(x-5)^2$
$f(x)=(2x-5)^3$		$f'(x)=6(2x-5)^2$
$f(x)=(5-2x)^3$		$f'(x)=-6(5-2x)^2$
$f(x)=(2x^2-5)^3$		$f'(x)=12x(2x^2-5)^2$
$f(x)=(2x^7-5)^3$		$f'(x)=42x^6(2x^7-5)^2$
$f(x)=(x-5)^{-3}$		$f'(x)=-3(x-5)^{-4}$
$f(x)=(2x-5)^{-3}$		$f'(x)=-6(2x-5)^{-4}$
$f(x)=(5-2x)^{-3}$		$f'(x)=6(5-2x)^{-4}$
$f(x)=(2x^2-5)^{-3}$		$f'(x)=-12x(2x^2-5)^{-4}$
$f(x)=(2x^7-5)^{-3}$		$f'(x)=-42x^6(2x^7-5)^{-4}$
$f(x)=\dfrac{1}{x-5}$ $=(x-5)^{-1}$		$f'(x)=\dfrac{-1}{(x-5)^2}$
$f(x)=\dfrac{1}{2x-5}$		$f'(x)=\dfrac{-2}{(2x-5)^2}$
$f(x)=\dfrac{1}{5-2x}$		$f'(x)=\dfrac{2}{(5-2x)^2}$
$f(x)=\dfrac{1}{2x^2-5}$		$f'(x)=\dfrac{-4x}{(2x^2-5)^2}$
$f(x)=\dfrac{1}{2x^7-5}$		$f'(x)=\dfrac{-14x^6}{(2x^7-5)^2}$

Function $f(x)$	**Derivative** $f'(x)$	**Compare with correct answer**
$f(x)=e^{\sin x}$		$f'(x)=e^{\sin x}\cos x$
$f(x)=\sin(\ln x)$		$f'(x)=\dfrac{\cos(\ln x)}{x}$
$f(x)=\sin^4 x$		$f'(x)=4\sin^3 x\cos x$
$f(x)=\cos(e^x)$		$f'(x)=-e^x\sin(e^x)$
$f(x)=3\cos^2 x$		$f'(x)=-6\cos x\sin x$
$f(x)=\ln\cos x$		$f'(x)=\dfrac{-\sin x}{\cos x}=-\tan x$
$f(x)=(1+e^x)^3$		$f'(x)=3e^x(1+e^x)^2$
$f(x)=\sqrt{x+e^x}$		$f'(x)=\dfrac{1+e^x}{2\sqrt{x+e^x}}$
$f(x)=\sqrt{\sin x}$		$f'(x)=\dfrac{\cos x}{2\sqrt{\sin x}}$
$f(x)=\dfrac{1}{\cos x}$		$f'(x)=\dfrac{\sin x}{\cos^2 x}$
$f(x)=\dfrac{3}{\sin^2 x}$		$f'(x)=-\dfrac{6\cos x}{\sin^3 x}$
$f(x)=e^{x\sin x}$		$f'(x)=e^{x\sin x}(\sin x+x\cos x)$
$f(x)=x^2e^{3x}$		$f(x)=2xe^{3x}+3x^2e^{3x}$

14. [Maximum mark: 3 per function]

Differentiate the following functions:

Function $f(x)$	**Derivative** $f'(x)$
$f(x)=2(x^2+5)^3$	
$f(x)=2e^{x^2+1}$	
$f(x)=7e^{-x}+8e^{\frac{x}{2}}$	
$f(x)=3\cos 3x+x$	
$f(x)=\ln(x^2+1)+2\cos(\frac{\pi}{2}x)$	
$f(x)=\sqrt{x^2+5}+\sqrt[3]{x}$	
$f(x)=(2x+5)^3+(2x+5)^2+2x+5$	
$f(x)=\sin^2 x+\cos^2 x$	
$y=e^{2x}+\ln(2x+3)$	
$y=\sqrt{x^5+2x+1}$	
$y=\dfrac{2x+1}{3x-5}$	
$y=\dfrac{2x+1}{(3x-5)^2}$	
$y=\ln\dfrac{2x+1}{(3x-5)^2}$	

15. [Maximum mark: 10]

Find $\frac{ds}{dt}$ for each of the following functions

$s = 2020\cos t$	
$s = \cos(2020t)$	
$s = \cos(t^{2020})$	
$s = \cos^{2020} t$	

16. [Maximum mark: 18]

Given that $f(1) = 2$ and $f'(1) = 4$, find the derivatives of the following functions at $x = 1$

(i) $y = f(x)^2$ (ii) $y = f(x)^3$ (iii) $y = \ln f(x)$

(iv) $y = f(x^2)$ (v) $y = f(x^3)$ (vi) $y = \sqrt{f(x)}$

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17. [Maximum mark: 10]

(a) Show that the derivative of $y = \tan x$ is $\dfrac{1}{\cos^2 x}$. [2]

(b) Hence, differentiate the functions

(i) $y = x \tan x$ (ii) $y = \tan 3x$

(iii) $y = \tan^2 x$ (iv) $y = \tan^3 x$ [8]

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18. [Maximum mark: 4]

Differentiate each of the following with respect to x.

(a) $y = x\sin 3x$ [2]

(b) $y = \dfrac{\ln x}{x}$ [2]

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19. [Maximum mark: 4]

The point P$\left(\dfrac{1}{2}, 0\right)$ lies on the graph of the curve of $y = \sin(2x-1)$.

Find the gradient of the tangent to the curve at P.

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20. [Maximum mark: 4]

Differentiate with respect to x: (i) $y = (x^2 + 1)^2$. (ii) $y = \ln(3x - 1)$

21. [Maximum mark: 4]

Differentiate with respect to x (i) $\sqrt{3 - 4x}$ (ii) $e^{\sin x}$

22. [Maximum mark: 5]

Let $f(x) = e^{\frac{x}{3}} + 5\cos^2 x$. Find $f'(x)$.

23. [Maximum mark: 6]

Let $f(x) = \cos 2x$ and $g(x) = \ln(3x - 5)$.

(a) Find $f'(x)$. [2]

(b) Find $g'(x)$. [2]

(c) Let $h(x) = f(x) \times g(x)$. Find $h'(x)$. [2]

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24. [Maximum mark: 6]

(a) Let $f(x) = e^{5x}$. Write down $f'(x)$. [2]

(b) Let $g(x) = \sin 2x$. Write down $g'(x)$. [2]

(c) Let $h(x) = e^{5x} \sin 2x$. Find $h'(x)$. [2]

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25. [Maximum mark: 6]

Let $f(x) = e^{-3x}$ and $g(x) = \sin\left(x - \frac{\pi}{3}\right)$.

(a) Write down (i) $f'(x)$; (ii) $g'(x)$. [2]

(b) Let $h(x) = e^{-3x}\sin\left(x - \frac{\pi}{3}\right)$. Find the exact value of $h'\left(\frac{\pi}{3}\right)$. [4]

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26. [Maximum mark: 6]

Let $f(x) = (2x+7)^3$ and $g(x) = \cos^2(4x)$. Find (i) $f'(x)$; (ii) $g'(x)$

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27. [Maximum mark: 6]

The population p of bacteria at time t is given by $p = 100e^{0.05t}$. Calculate

(a) the value of p when $t = 0$; [2]

(b) the rate of increase of the population when $t = 10$. [4]

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28. [Maximum mark: 8]

The number of bacteria, n, in a dish, after t minutes is given by $n = 800e^{013t}$.

(a) Find the value of n when $t = 0$. [2]

(b) Find the rate at which n is increasing when $t = 15$. [2]

(c) After k minutes, the rate of increase in n is greater than 10 000 bacteria per minute. Find the least value of k, where $k \in \mathbb{Z}$. [4]

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29. [Maximum mark: 8]

Consider the curve $y = \ln(3x - 1)$. Let P be the point on the curve where $x = 2$.

(a) Write down the gradient of the curve at P. [2]

(b) Find the equation of the tangent to the curve at P. [2]

(c) The normal to the curve at P cuts the x-axis at R. Find the coordinates of R. [4]

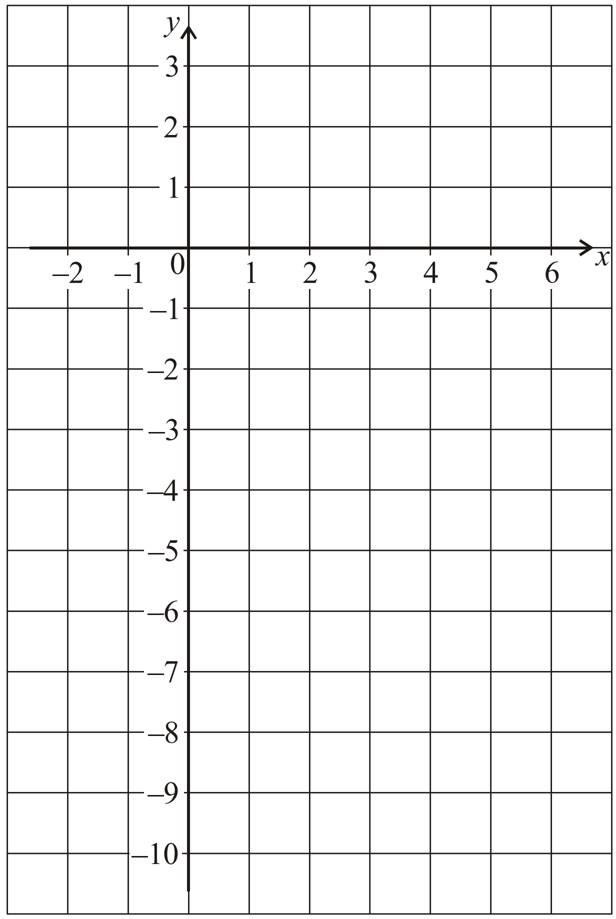
30. [Maximum mark: 7]

Let $f(x) = 3x - e^{x-2} - 4$, for $-1 \le x \le 5$.

(a) Find the x-intercepts of the graph of f. [3]

(b) On the grid below, sketch the graph of f. [3]

(c) Write down the gradient of the graph of f at $x = 2$. [1]



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31. [Maximum mark: 6]

If $y = \ln(2x-1)$ find $\frac{d^2y}{dx^2}$.

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32. [Maximum mark: 6]

Consider the function $y = \tan x - 8\sin x$.

(a) Find $\frac{dy}{dx}$. [2]

(b) Find the value of $\cos x$ for which $\frac{dy}{dx} = 0$. [2]

(c) Solve the equation $\frac{dy}{dx} = 0$. for $-\pi \le x \le 2\pi$. [2]

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33. [Maximum mark: 6]

Let $y = e^{3x}\sin(\pi x)$. Find

(a) $\frac{\mathrm{d}y}{\mathrm{d}x}$. [4]

(b) the smallest positive value of *x* for which $\frac{\mathrm{d}y}{\mathrm{d}x} = 0$. [2]

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34. [Maximum mark: 6]

Let $f(x) = \cos^3(4x+1)$, $0 \le x \le 1$. Find

(a) $f'(x)$ [3]

(b) the **exact** values of the three roots of $f'(x) = 0$. [3]

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35. [Maximum mark: 6]

Let f be a cubic polynomial function. Given that $f(0)=2$, $f'(0)=-3$, $f(1)=f'(1)$ and $f''(-1)=6$, find $f(x)$.

36. [Maximum mark: 3 per function]

The following table shows the values of two functions f and g and their derivatives when $x=1$ and $x=0$.

x	$f(x)$	$f'(x)$	$g(x)$	$g'(x)$
0	4	1	-4	5
1	-2	3	-1	2

Find the derivatives of the following functions when $x=1$.

$3f(x)+5g(x)$	
$f(x)g(x)$	
$\dfrac{f(x)}{g(x)}$	
$f(x)^3$	
$\ln f(x)$	
$f(\ln x)$	
$e^{f(x)}$	
$f(e^{x-1})$	
$f(x-1)$	
$f(2x-2)$	
$f(-g(x))$	
$g(f(x)+2)$	

B. Paper 2 questions (LONG)

37. [Maximum mark: 10]

Let $f(x)=1+3\cos(2x)$ for $0 \le x \le \pi$, and x is in radians.

(a) (i) Find $f'(x)$.

(ii) Find the values for x for which $f'(x)=0$; Give your answers in terms of π. [6]

The function $g(x)$ is defined as $g(x)=f(2x)-1,\ 0 \le x \le \dfrac{\pi}{2}$.

(b) (i) The graph of f may be transformed to the graph of g by a stretch in the x-direction with scale factor $\frac{1}{2}$ followed by another transformation.

Describe fully this other transformation.

(ii) Find the solution to the equation $g(x)=f(x)$. [4]

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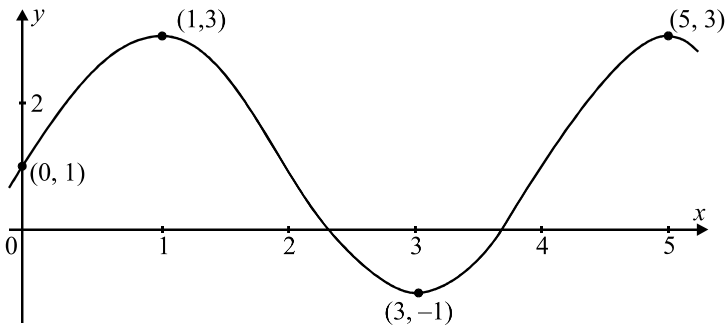
38. [Maximum mark: 20]

The diagram shows the graph of the function f given by $f(x) = A\sin\left(\frac{\pi}{2}x\right) + B$, for $0 \le x \le 5$, where A and B are constants, and x is measured in radians.



The graph includes the points (1, 3) and (5, 3), which are maximum points of the graph.

(a) Show that $A = 2$, and find the value of B. [5]

(b) Show that $f'(x) = \pi\cos\left(\frac{\pi}{2}x\right)$. [4]

The line $y = k - \pi x$ is a tangent line to the graph for $0 \le x \le 5$.

(c) Find

(i) the point where this tangent meets the curve;

(ii) the value of k. [6]

(d) Solve the equation $f(x) = 2$ for $0 \le x \le 5$. [5]

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39. [Maximum mark: 14]

The following diagram shows a waterwheel with a bucket. The wheel rotates at a constant rate in an anticlockwise (counterclockwise) direction.

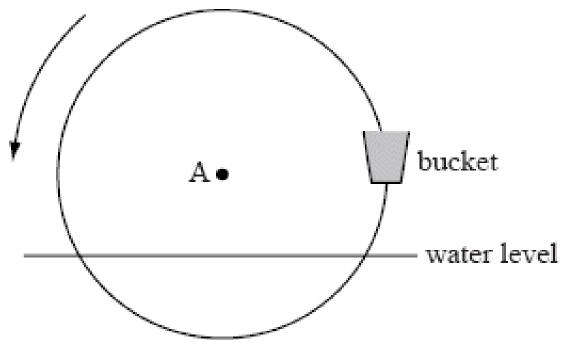

diagram not to scale

The diameter of the wheel is 8 metres. The centre of the wheel, A, is 2 metres above the water level. After t seconds, the height of the bucket above the water level is given by $h = a \sin bt + 2$.

(a) Show that $a = 4$. [2]

The wheel turns at a rate of one rotation every 30 seconds.

(b) Show that $b = \frac{\pi}{15}$. [2]

In the first rotation, there are two values of t when the bucket is **descending** at a rate of 0.5 m s^{-1}.

(c) Find these values of t. [6]

(d) Determine whether the bucket is underwater at the second value of t. [4]

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40. [Maximum mark: 18]

Let $f(x) = 3\sin x + 4\cos x$, for $-2\pi \leq x \leq 2\pi$.

(a) Sketch the graph of f. [3]

(b) Write down

(i) the amplitude; (ii) the period; (iii) the x-intercept between $-\frac{\pi}{2}$ and 0. [3]

(c) Hence write $f(x)$ in the form $p\sin(qx + r)$. [3]

(d) Write down one value of x such that $f'(x) = 0$. [2]

(e) Write down the two values of k for which the equation $f(x) = k$ has exactly two solutions. [2]

(f) Let $g(x) = \ln(x + 1)$, for $0 \leq x \leq \pi$. There is a value of x, between 0 and 1, for which the gradient of f is equal to the gradient of g. Find this value of x. [5]

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41. [Maximum mark: 15]

(a) The function g is defined by $g(x) = \frac{e^x}{\sqrt{x}}$, for $0 < x \leq 3$.

(i) Sketch the graph of g.

(ii) Find $g'(x)$.

(iii) Write down an expression representing the gradient of the normal to the curve at any point. [8]

(b) Let P be the point (x, y) on the graph of g, and Q the point (1,0).

(i) Find the gradient of (PQ) in terms of x.

(ii) Given that the line (PQ) is a normal to the graph of g at the point P, find the minimum distance from the point Q to the graph of g. [7]

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