INTERNATIONAL BACCALAUREATE

Mathematics: applications and interpretation

MAI

EXERCISES [MAI 1.12-1.13]

MATRICES

Compiled by Christos Nikolaidis

A. Paper 1 questions (SHORT)

MATRIX OPERATIONS

1. [Maximum mark: 12]

For each of the following operations state whether it is possible or not. If so, find the result.

$\begin{pmatrix} 3 & 3-x \\ 2-y & 1 \end{pmatrix} + \begin{pmatrix} 2 & x \\ 1+y & 0 \end{pmatrix}$	
$-2\begin{pmatrix} 0 & 1 \\ -4 & 2 \\ 4 & 5 \end{pmatrix} + 3\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 \\ 2 \end{pmatrix} + \begin{pmatrix} 0 & 0 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$	
$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$	

2. [Maximum mark: 8]

(a) For each of the following operations state whether it is possible or not.
If so, find the result.

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	

[5]

(b) Let $\boldsymbol{A}$ be a 2×3 matrix. Given that $\boldsymbol{I}_2$ and $\boldsymbol{I}_3$ are the identity 2×2 and 3×3 matrices respectively, write down **three** properties that describe the results in question (a). [3]

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3. [Maximum mark: 6]

(a) Let $\begin{pmatrix} b & 3 \\ 7 & 8 \end{pmatrix} + \begin{pmatrix} 9 & 5 \\ -2 & 7 \end{pmatrix} = \begin{pmatrix} 4 & 8 \\ a & 15 \end{pmatrix}$.

(i) Write down the value of a. (ii) Find the value of b. [4]

(b) Let $3\begin{pmatrix} -4 & 8 \\ 2 & 1 \end{pmatrix} - 5\begin{pmatrix} 2 & 0 \\ q & -4 \end{pmatrix} = \begin{pmatrix} -22 & 24 \\ 9 & 23 \end{pmatrix}$. Find the value of q. [2]

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4. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 & 0 \\ -2 & 1 \end{pmatrix}$.

Find (i) $\boldsymbol{A} + \boldsymbol{B}$; (ii) $-3\boldsymbol{A}$; (iii) $\boldsymbol{AB}$.

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5. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 1 & 0 \\ d & e \end{pmatrix}$. Giving your answers in terms of a, b, c, d and e,

(a) write down $\boldsymbol{A} + \boldsymbol{B}$; [2]

(b) find $\boldsymbol{AB}$. [3]

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6. [Maximum mark: 6]

Given that $\begin{pmatrix} 2x & -y \\ 3z & w \end{pmatrix} = \begin{pmatrix} x & -4 \\ -1 & 2w \end{pmatrix} + \begin{pmatrix} 4 & x+y \\ z+w & 3 \end{pmatrix}$, find the values of x, y, z, w.

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7. [Maximum mark: 6]

Let $A = \begin{pmatrix} 1 & 0 & 2 \\ 3 & 5 & -4 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 4 & 7 \\ 2 & 5 & 8 \\ 3 & 6 & 9 \end{pmatrix}$, $B_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$, $B_2 = \begin{pmatrix} 4 \\ 5 \\ 6 \end{pmatrix}$, $B_3 = \begin{pmatrix} 7 \\ 8 \\ 9 \end{pmatrix}$.

(a) Find AB. [2]

(b) Find AB_1, AB_2, AB_3. What do you observe? [4]

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8. [Maximum mark: 5]

Let $A = \begin{pmatrix} 1 & 3 & 1 \\ 2 & 1 & 4 \\ 4 & 7 & 6 \end{pmatrix}$ and $B = \begin{pmatrix} x & -2 & 3 \\ 1 & 1 & 1 \\ 4 & x & 2 \end{pmatrix}$. Show that $AB \neq BA$ holds for any $x \in R$.

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9. [Maximum mark: 7]

Let $A = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$. Find all matrices B such that $AB = BA$ in the following cases:

(i) if $B = \begin{pmatrix} 1 & 0 \\ 0 & x \end{pmatrix}$ (ii) if $B = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$

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10. [Maximum mark: 10]

Let $A=\begin{pmatrix}-1 & 3 & 5\\ 1 & -3 & -5\\ -1 & 3 & 5\end{pmatrix}$, $B=\begin{pmatrix}0 & 1 & -1\\ 4 & -3 & 4\\ 3 & -3 & 4\end{pmatrix}$, $C=\begin{pmatrix}1 & 1 & 3\\ 5 & 2 & 6\\ -2 & -1 & -3\end{pmatrix}$

(a) Find A^2; What do you observe? **Hence** deduce a formula for A^n, $n \in Z^+$. [3]

(b) Find B^2; What do you observe? **Hence** deduce a formula for B^n, $n \in Z^+$. [4]

(c) Find C^3; What do you observe? **Hence** deduce a formula for C^n, $n \geq 3$. [3]

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11. [Maximum mark: 6]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix}$.

(a) Find $\boldsymbol{AB}$. [3]

(b) The matrix $\boldsymbol{C} = \begin{pmatrix} 20 \\ 28 \end{pmatrix}$ and $2\boldsymbol{AB} = \boldsymbol{C}$. Find the value of x. [3]

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12. [Maximum mark: 5]

If $\boldsymbol{A} = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}$ and $\boldsymbol{A}^2$ is a matrix whose entries are all 0, find k.

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13. [Maximum mark: 6]

If $\boldsymbol{A} = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}$, find two values of x and y, given that $\boldsymbol{AB} = \boldsymbol{BA}$.

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14. [Maximum mark: 5]

Let $\boldsymbol{W} = \begin{pmatrix} 1 & 3 & 2 \\ 2 & 0 & 1 \\ 0 & 1 & 3 \end{pmatrix}$, $\boldsymbol{P} = \begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$ and $\boldsymbol{Q} = \begin{pmatrix} 26 \\ 12 \\ 10 \end{pmatrix}$

(i) Find $\boldsymbol{WP}$. (ii) Given that $2\boldsymbol{WP} + \boldsymbol{S} = \boldsymbol{Q}$, find $\boldsymbol{S}$.

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DETERMINANT AND INVERSE MATRIX

15. [Maximum mark: 14]

For each of the following matrices, use the appropriate formulas to find its determinant and its inverse (it there exists).

Matrix	Determinant	Inverse
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$		
$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$		
$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$		
$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$		
$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$		
$E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$		
$F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$		

16. [Maximum mark: 6]

For each of the following matrices, use the appropriate formulas to find its determinant and its inverse (it there exists).

Matrix	Determinant	Inverse
$A = \begin{pmatrix} 2 & 5 \\ -2 & 7 \end{pmatrix}$		
$B = \begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$		
$C = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$		

17. [Maximum mark: 4]

If $A = \begin{pmatrix} 2p & 3 \\ -4p & p \end{pmatrix}$ and $\det A = 14$, find the possible values of p.

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18. [Maximum mark: 4]

Find the values of the real number k for which the determinant of the matrix

$\begin{pmatrix} k-4 & 3 \\ -2 & k+1 \end{pmatrix}$ is equal to zero.

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19. [Maximum mark: 5]

(a) Given that $a \neq 0$, $b \neq 0$, show that $A = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}$ and $B = \begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix}$ are inverse to each other. [3]

(b) **Hence**, write down the inverse of $M = \begin{pmatrix} 3 & 0 \\ 0 & 1/5 \end{pmatrix}$. [2]

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20. [Maximum mark: 7]

Let $A = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 3 \\ 1 & 2 & 4 \end{pmatrix}$ and $B = \begin{pmatrix} 6 & -2 & -3 \\ -1 & 1 & x \\ -1 & x & 1 \end{pmatrix}$.

(a) Write down the value of $\det A$. [1]

(b) Find the value of x, given that $AB = I = BA$. [3]

(c) Write down A^{-1} and B^{-1}. [2]

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21. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 2 \\ k & 4 \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix}$. Find, in terms of k,

(a) $2\boldsymbol{A} - \boldsymbol{B}$; [3]

(b) $\det(2\boldsymbol{A} - \boldsymbol{B})$. [2]

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22. [Maximum mark: 6]

Let $\boldsymbol{C} = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}$ and $\boldsymbol{D} = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}$.

(a) Find $\boldsymbol{CD}$. [2]

(b) Find $\boldsymbol{D}^{-1}$. [2]

(c) Find $\boldsymbol{D}^2$. [2]

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23. [Maximum mark: 6]

Let $\boldsymbol{A} = \begin{pmatrix} 1 & -2 \\ 3 & p \end{pmatrix}$ and $\boldsymbol{B} = \begin{pmatrix} -2 & 1 \\ q & \frac{1}{2} \end{pmatrix}$.

(a) Find $\boldsymbol{AB}$ in terms of p and q. [3]

(b) Matrix $\boldsymbol{B}$ is the inverse of matrix $\boldsymbol{A}$. Find the value of p and of q. [3]

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24. [Maximum mark: 6]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & x \\ -2 & -3 \end{pmatrix}$.

(a) Find the value of x for which $\boldsymbol{A}^{-1}$ does not exist. [3]

(b) Given that $\boldsymbol{A} = \boldsymbol{A}^{-1}$, find x. [3]

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25. [Maximum mark: 4]

The square matrix $\boldsymbol{X}$ is such that $\boldsymbol{X}^3 = 0$. Show that the inverse of the matrix $(\boldsymbol{I} - \boldsymbol{X})$ is $\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2$.

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26. [Maximum mark: 5]

Let $\boldsymbol{A} = \begin{pmatrix} 3 & 1 & 2 \\ 9 & 5 & 8 \\ 7 & 4 & 6 \end{pmatrix}$.

(a) Write down the determinant of $\boldsymbol{A}$. [1]

(b) Write down the inverse of $\boldsymbol{A}$. [1]

(c) By investigating the determinant of $\boldsymbol{A}^n$, for several values of n, write down a formula for $\det \boldsymbol{A}^n$ in terms of n. [3]

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B. Paper 2 questions (*LONG*)

27. [Maximum mark: 10]

The following properties are wrong. Write down the correct answer.

	Wrong answer	Correct answer
$\boldsymbol{AX}+\boldsymbol{BX}=$	$\boldsymbol{X}(\boldsymbol{A}+\boldsymbol{B})$	
$\boldsymbol{XA}+\boldsymbol{XB}=$	$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}$	
$\boldsymbol{AB}+\boldsymbol{A}=$	$\boldsymbol{A}(\boldsymbol{B}+1)$	
$\boldsymbol{AB}+2\boldsymbol{A}=$	$(\boldsymbol{B}+2\boldsymbol{I})\boldsymbol{A}$	
$\boldsymbol{BA}+k\boldsymbol{A}=$	$(\boldsymbol{B}+k)\boldsymbol{A}$	
$\boldsymbol{A}^2+\boldsymbol{A}=$	$\boldsymbol{A}(\boldsymbol{A}+1)$	
$(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A}=$	$\boldsymbol{AB}+\boldsymbol{AC}$	
$(\boldsymbol{A}+\boldsymbol{C})\boldsymbol{A}=$	$\boldsymbol{A}^2+\boldsymbol{AC}$	
$(\boldsymbol{A}+\boldsymbol{B})^2=$	$\boldsymbol{A}^2+2\boldsymbol{AB}+\boldsymbol{B}^2$	
$(\boldsymbol{A}+\boldsymbol{I})^2=$	$\boldsymbol{A}^2+2\boldsymbol{A}+1$	

28. [Maximum mark: 15]

Let $A=\begin{pmatrix}1 & 2\\ 0 & 1\end{pmatrix}$

(a) Find the matrices A^2, A^3, A^4. [3]

(b) Guess a formula for the matrix A^n, for any $n \in Z^+$. [2]

(c) Find A^{10} by using your GDC. Confirm that your guess is true for $n=10$. [2]

(d) Write down the result of your guess for $n=0$. What do you notice? [2]

In fact, the formula of your guess in (b) can be extended for any $n \in Z$ (that is for negative powers as well)

(e) Find the inverse matrix A^{-1}. Does the result agree with your guess? [2]

(f) Write down the matrix A^{-n}, according to your guess.

Show that A^{-n} is the inverse of A^n by multiplying the two matrices. [4]

29. [Maximum mark: 14]

Let S_n be the sum of the first n terms of the arithmetic series $2 + 4 + 6 + \ldots$.

(a) Find (i) S_4; (ii) S_{100}. [4]

Let $\boldsymbol{M} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix}$.

(b) Find $\boldsymbol{M}^2$ and $\boldsymbol{M}^3$. [3]

It may now be assumed that $\boldsymbol{M}^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$, for $n \geq 4$.

The sum $\boldsymbol{T}_n$ is defined by $\boldsymbol{T}_n = \boldsymbol{M}^1 + \boldsymbol{M}^2 + \boldsymbol{M}^3 + \ldots + \boldsymbol{M}^n$.

(c) (i) Write down $\boldsymbol{M}^4$. (ii) Find $\boldsymbol{T}_4$. [4]

(d) Using your results from part (a) (ii), find $\boldsymbol{T}_{100}$. [3]

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