EXERCISES [MAI 5.6-5.7]

RULES OF DIFFERENTIATION

SOLUTIONS

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A. Paper 1 questions (SHORT)

BASIC RULES OF DIFFERENTIATION

1.

$y = 7x^3 - 2 - 5e^x - 3\sin x$	$y' = 21x^2 - 5e^x - 3\cos x$
$y = \sqrt{x} + \ln x$	$y' = \dfrac{1}{2\sqrt{x}} + \dfrac{1}{x}$
$y = \sqrt{x}\ln x$	$y' = \dfrac{\ln x}{2\sqrt{x}} + \dfrac{\sqrt{x}}{x}$
$y = \dfrac{\ln x}{\sqrt{x}}$	$y' = \dfrac{\sqrt{x}}{x^2} - \dfrac{\ln x}{2x\sqrt{x}}$
$y = \dfrac{2x+1}{3x-5}$	$y' = \dfrac{-13}{(3x-5)^2}$
$y = x + ex + \ln\pi$	$y' = 1 + e$
$y = x^2 + \ln x + x^2\ln x$	$y' = 2x + \dfrac{1}{x} + 2x\ln x + x = 3x + \dfrac{1}{x} + 2x\ln x$
$y = x\sin x\ln x$	$y' = \sin x\ln x + x\cos x\ln x + \sin x$
$y = x^2e^x\ln x$	$y = 2xe^x\ln x + x^2e^x\ln x + xe^x$

2. (a) $f'(x) = 6x^2 + \dfrac{1}{x}$ (b) $f'(1) = 6 + 1 = 7$

3. (a) $f'(x) = \dfrac{3x^2\sin x - (x^3+1)\cos x}{\sin^2 x}$

(b) Directly by GDC (i) $f'(\frac{\pi}{4}) \cong 0.518$ (ii) $f'(1) \cong 2.04$

[Notice: the exact value for (i) is $f'(\frac{\pi}{4}) = \dfrac{3\pi^2}{16}\sqrt{2} - \dfrac{\pi^3+64}{64}\sqrt{2}$]

4. (a) $\dfrac{dy}{dx} = 2f'(x) - 3g'(x)$, at $x = 1$ the value is –7

(b) $\dfrac{dy}{dx} = f'(x)g(x) + f(x)g'(x)$, at $x = 1$ the value is 22

(c) $\dfrac{dy}{dx} = \dfrac{f'(x)g(x) - f(x)g'(x)}{g(x)^2}$, at $x = 1$ the value is $\dfrac{2}{9}$

(d) $\dfrac{dy}{dx} = 6x^2 + 5f'(x)$, at $x = 1$ the value is 26

5. $f(x)=6x^{\frac{2}{3}}$, $f'(x)=4x^{-\frac{1}{3}}\left(=\frac{4}{x^{\frac{1}{3}}}=\frac{4}{\sqrt[3]{x}}\right)$

6. **METHOD 1 (quotient)**

$$h'(x)=\frac{(\cos x)(6)-(6x)(-\sin x)}{(\cos x)^2}$$

$$h'(0)=\frac{(\cos 0)(6)-(6\times 0)(-\sin 0)}{(\cos 0)^2}=6$$

METHOD 2 (product)

$h(x) = 6x \times (\cos x)^{-1}$

$h'(x) = (6x)\,(-(\cos x)^{-2}\,(-\sin x)) + (6)\,(\cos x)^{-1}$

$h'(0) = (6\times 0)\,(-(\cos 0)^{-2}\,(-\sin 0)) + (6)\,(\cos 0)^{-1} = 6$

7. (a) $g'(x) = 2\sin x + 2x\cos x$

(b) $g'(\pi) = 2\sin\pi + 2\pi\cos\pi = -2\pi$

8. (a) $f'(x) = k\cos x + 3$

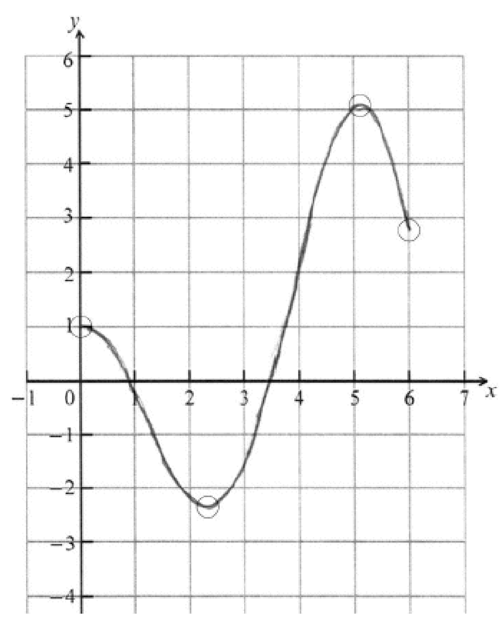
(b) $k\cos\left(\frac{\pi}{3}\right) + 3 = 8 \Rightarrow k\left(\frac{1}{2}\right) + 3 = 8 \Rightarrow k = 10$

9. (a) $x=\frac{1}{5}$

(b) $f'(x)=\frac{(5x-1)(6x)-(3x^2)(5)}{(5x-1)^2}=\frac{30x^2-6x-15x^2}{(5x-1)^2}=\frac{15x^2-6x}{(5x-1)^2}$

10. (a) $f'(x) = \cos x - x\sin x$

(b)



Mind: *correct domain, $0 \le x \le 6$ with endpoints in circles,*
approximately correct shape, local min and local max in circles.

(c) $y\in[-2.38, 5.10]$

11. (a) $f'(x) = e^x \times (-\sin x) + \cos x \times e^x = e^x \cos x - e^x \sin x$

$f'(\pi) = e^\pi \cos \pi - e^\pi \sin \pi = -e^\pi$

gradient of normal $= \dfrac{1}{e^\pi}$

(b) $f'\left(\dfrac{\pi}{4}\right) = 0$

12. (a) Point (0,0), $f'(x) = e^x + xe^x$, $m_T = 1$, Tangent line $y = x$

(b) $m_N = -1$, Normal line $y = -x$

(c) $e^x + xe^x = 0 \Leftrightarrow e^x(1+x) = 0 \Leftrightarrow x = -1$

CHAIN RULE

13. Solutions are shown in the question.

14.

Function $f(x)$	**Derivative** $f'(x)$
$f(x) = 2(x^2+5)^3$	$f'(x) = 12x(x^2+5)^2$
$f(x) = 2e^{x^2+1}$	$f'(x) = 4xe^{x^2+1}$
$f(x) = 7e^{-x} + 8e^{\frac{x}{2}}$	$f'(x) = -7e^{-x} + 4e^{\frac{x}{2}}$
$f(x) = 3\cos 3x + x$	$f'(x) = -9\sin 3x + 1$
$f(x) = \ln(x^2+1) + 2\cos(\frac{\pi}{2}x)$	$f'(x) = \dfrac{2x}{x^2+1} - \pi \sin(\frac{\pi}{2}x)$
$f(x) = \sqrt{x^2+5} + \sqrt[3]{x}$	$f'(x) = \dfrac{x}{\sqrt{x^2+5}} + \dfrac{1}{3}x^{-2/3}$
$f(x) = (2x+5)^3 + (2x+5)^2 + 2x + 5$	$f'(x) = 6(2x+5)^2 + 8x + 22$
$f(x) = \sin^2 x + \cos^2 x$	$f'(x) = 0$
$y = e^{2x} + \ln(2x+3)$	$f'(x) = 2e^{2x} + \dfrac{2}{2x+3}$
$y = \sqrt{x^5+2x+1}$	$f'(x) = \dfrac{5x^4+2}{2\sqrt{x^5+2x+1}}$
$y = \dfrac{2x+1}{3x-5}$	$f'(x) = \dfrac{2(3x-5)-3(2x+1)}{(3x-5)^2} = \dfrac{-13}{(3x-5)^2}$
$y = \dfrac{2x+1}{(3x-5)^2}$ $(= (2x+1)(3x-5)^{-2})$	$f'(x) = 2(3x-5)^{-2} - 6(2x+1)(3x-5)^{-3}$
$y = \ln\dfrac{2x+1}{(3x-5)^2}$ $(= \ln(2x+1) - 2\ln(3x-5))$	$f'(x) = \dfrac{2}{2x+1} - \dfrac{6}{3x-5}$

15.

$s = 2020\cos t$	$ds/dt = -2020\sin t$
$s = \cos(2020t)$	$ds/dt = -2020\sin 2020t$
$s = \cos(t^{2020})$	$ds/dt = -2020t^{2019}\sin(t^{2020})$
$s = \cos^{2020} t$	$ds/dt = -2020(\sin t)(\cos^{2019} t)$

16. (i) $y = f(x)^2 \Rightarrow \dfrac{dy}{dx} = 2f(x)f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = 2\times2\times4 = 16$

(ii) $y = f(x)^3 \Rightarrow \dfrac{dy}{dx} = 3f(x)^2 f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = 3\times2^2\times4 = 48$

(iii) $y = \ln f(x) \Rightarrow \dfrac{dy}{dx} = \dfrac{f'(x)}{f(x)}$. At $x = 1$, $\dfrac{dy}{dx} = \dfrac{4}{2} = 2$

(iv) $y = f(x^2) \Rightarrow \dfrac{dy}{dx} = f'(x^2)\times 2x$. At $x = 1$, $\dfrac{dy}{dx} = 2\times2\times1 = 4$

(v) $y = f(x^3) \Rightarrow \dfrac{dy}{dx} = f'(x^3)\times 3x^2$. At $x = 1$, $\dfrac{dy}{dx} = 2\times3\times1^2 = 6$

(vi) $y = \sqrt{f(x)} \Rightarrow \dfrac{dy}{dx} = \dfrac{1}{2\sqrt{f(x)}}f'(x)$. At $x = 1$, $\dfrac{dy}{dx} = \dfrac{1}{2\sqrt{2}}\times4 = \dfrac{2}{\sqrt{2}} = \sqrt{2}$

17. (a) $y = \tan x = \dfrac{\sin x}{\cos x}$

$$\frac{dy}{dx} = \frac{\sin x\sin x - \cos x(-\cos x)}{\cos^2 x} = \frac{\sin^2 x + \cos^2 x}{\cos^2 x} = \frac{1}{\cos^2 x}$$

(b) (i) $\dfrac{dy}{dx} = \dfrac{x}{\cos^2 x} + \tan x$

(ii) $\dfrac{dy}{dx} = \dfrac{3}{\cos^2 3x}$

(iii) $\dfrac{dy}{dx} = 2\tan x\dfrac{1}{\cos^2 x}\left(= \dfrac{2\sin x}{\cos^3 x}\right)$

(iv) $\dfrac{dy}{dx} = 3\tan^2 x\dfrac{1}{\cos^2 x}\left(= \dfrac{3\sin^2 x}{\cos^4 x}\right)$

18. (i) $\dfrac{dy}{dx} = \sin 3x + 3x\cos 3x$

(i) $\dfrac{dy}{dx} = \dfrac{x\times\frac{1}{x} - \ln x}{x^2} = \dfrac{1-\ln x}{x^2}$

19. $y = \sin(2x - 1)$ $\quad \dfrac{dy}{dx} = 2\cos(2x - 1)$

At $\left(\dfrac{1}{2}, 0\right)$, the gradient of the tangent = 2 cos 0 = 2

20. (a) $\frac{\mathrm{d}}{\mathrm{d}x}(x^2+1)^2 = 2(x^2+1) \times (2x) = 4x(x^2+1)$

(b) $\frac{\mathrm{d}}{\mathrm{d}x}(\ln(3x-1)) = \frac{1}{3x-1} \times (3) = \frac{3}{3x-1}$

21. (a) $\frac{\mathrm{d}y}{\mathrm{d}x} = \frac{-4}{2\sqrt{3-4x}} = \frac{-2}{\sqrt{3-4x}}$

OR $y = \sqrt{3-4x} = (3-4x)^{\frac{1}{2}}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = \frac{1}{2}(3-4x)^{-\frac{1}{2}}(-4)$

(b) $y = \mathrm{e}^{\sin x}$ $\quad \frac{\mathrm{d}y}{\mathrm{d}x} = (\cos x)(\mathrm{e}^{\sin x})$

22. $f'(x) = \frac{1}{3}\mathrm{e}^{\frac{x}{3}} - 10\cos x \sin x$

23. (a) $f'(x) = -\sin 2x \times 2$ $(= -2\sin 2x)$

(b) $g'(x) = 3 \times \frac{1}{3x-5}\left(= \frac{3}{3x-5}\right)$

(c) product rule: $h'(x) = (\cos 2x)\left(\frac{3}{3x-5}\right) + \ln(3x-5)(-2\sin 2x)$

24. (a) $f'(x) = 5\mathrm{e}^{5x}$

(b) $g'(x) = 2\cos 2x$

(c) $h' = fg' + gf' = \mathrm{e}^{5x}(2\cos 2x) + \sin 2x\,(5\mathrm{e}^{5x})$

25. (a) (i) $-3\mathrm{e}^{-3x}$ (ii) $\cos\left(x - \frac{\pi}{3}\right)$

(b) $h'(x) = -3\mathrm{e}^{-3x}\sin\left(x - \frac{\pi}{3}\right) + \mathrm{e}^{-3x}\cos\left(x - \frac{\pi}{3}\right)$

$h'\left(\frac{\pi}{3}\right) = -3\mathrm{e}^{-3\frac{\pi}{3}}\sin\left(\frac{\pi}{3} - \frac{\pi}{3}\right) + \mathrm{e}^{-3\frac{\pi}{3}}\cos\left(\frac{\pi}{3} - \frac{\pi}{3}\right) = \mathrm{e}^{-\pi}$

26. (a) $f'(x) = 3(2x+7)^2 \times 2 = 6(2x+7)^2$ $\quad (= 24x^2 + 168x + 294)$

(b) $g'(x) = 2\cos(4x)(-\sin(4x))(4) = -8\cos(4x)\sin(4x)$ $\quad (= -4\sin(8x))$

27. (a) $p = 100\mathrm{e}^0 = 100$

(b) Rate of increase is $\frac{\mathrm{d}p}{\mathrm{d}t} = 0.05 \times 100\mathrm{e}^{0.05t} = 5\mathrm{e}^{0.05t}$

When $t = 10$ $\quad \frac{\mathrm{d}p}{\mathrm{d}t} = 5\mathrm{e}^{0.05(10)} = 5\mathrm{e}^{0.5}$ $\quad \left(= 8.24 = 5\sqrt{\mathrm{e}}\right)$

28. (a) $n = 800\mathrm{e}^0 = 800$

(b) derivative: $n'(15) = 731$

(c) **METHOD 1**
setting up inequality. $n'(t) > 10\,000$

$k = 35.1226...$, least value of k is 36

METHOD 2
$n'(35) = 9842$, **and** $n'(36) = 11208$
least value of k is 36

29. (a) gradient is 0.6

(b) $y - \ln 5 = 0.6\,(x - 2)$ **OR** directly by GDC: $y = 0.6\,x + 0.409$

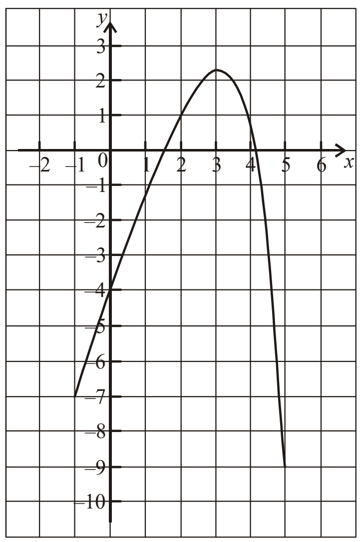
(c) at $x = 2$, $y = \ln 5$ (= 1.609...)
gradient of normal = – 5/3 = –1.6666...

normal: $y - \ln 5 = -\dfrac{5}{3}(x - 2)$ **OR** directly by GDC $y = -\,1.66666x + 4.94277$

For $y = 0$: $x = 2.97$ (accept 2.96)

coordinates of R are (2.97, 0)

30. (a) (1.54, 0) (4.13, 0) (accept $x = 1.54$ $x = 4.13$)
(b)



Note: *Curve passing through (0, – 4), a range of approximately –9 to 2.3.*

(c) gradient is 2

31. $y = \ln(2x - 1) \Rightarrow \dfrac{dy}{dx} = \dfrac{2}{2x-1} \Rightarrow \dfrac{dy}{dx} = 2(2x-1)^{-1}$

$$\Rightarrow \frac{d^2y}{dx^2} = -\,2(2x-1)^{-2}\,(2)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{-\,4}{(2x-1)^2}$$

32. (a) $\dfrac{dy}{dx} = \dfrac{1}{\cos^2 x} - 8\cos x$

(b) $\dfrac{dy}{dx} = \dfrac{1 - 8\cos^3 x}{\cos^2 x} = 0 \Rightarrow \cos x = \dfrac{1}{2}$

(c) $x = -\dfrac{\pi}{3}$, $x = \dfrac{\pi}{3}$, $x = \dfrac{5\pi}{3}$

33. $y = e^{3x}\sin(\pi x)$

(a) $\dfrac{dy}{dx} = 3e^{3x}\sin(\pi x) + \pi e^{3x}\cos(\pi x)$

(b) $x = 0.7426\ldots$ (0.743 to 3 s.f.)

34. (a) $-12\cos^2(4x+1)\sin(4x+1)$

(b) $x = \dfrac{\pi}{8} - \dfrac{1}{4}$, $x = \dfrac{3\pi}{8} - \dfrac{1}{4}$, $x = \dfrac{\pi - 1}{4}$

35.

$f(x) = ax^3 + bx^2 + cx + d$

$f'(x) = 3ax^2 + 2bx + c$

$f''(x) = 6ax + 2b$

$f(0) = 2 = d$

$f'(1) = f(1) \rightarrow a + b + c + 2 = 3a + 2b + c$

$2 = 2a + b$

$f'(0) = -3 = c$

$f''(-1) = 6 = -6a + 2b$

$b = \dfrac{12}{5}, a = -\dfrac{1}{5}$

$f(x) = -\dfrac{1}{5}x^3 + \dfrac{12}{5}x^2 - 3x + 2$ (Accept $a = -\dfrac{1}{5}, b = \dfrac{12}{5}, c = -3, d = 2$)

36.

$3f(x) + 5g(x)$	19
$f(x)g(x)$	-7
$f(x)/g(x)$	1
$f(x)^3$	36
$\ln f(x)$	-3/2
$f(\ln x)$	1
$e^{f(x)}$	3/e2
$f(e^{x-1})$	3
$f(x-1)$	1
$f(2x-2)$	2
$f(-g(x))$	-6
$g(f(x)+2)$	15

For the last 2:

$f'(-g(x)) \times (-g'(x))$. Hence, at $x = 1$, $f'(-g(1)) \times (-g'(1)) = f'(1) \times (-2) = -6$

$g'(f(x)+2) \times (f'(x))$. Hence, at $x = 1$, $g'(f(1)+2) \times (f'(1)) = g'(0) \times 3 = 15$

B. Paper 2 questions (*LONG*)

37. (a) (i) $f'(x) = -6\sin 2x$

(ii) **EITHER** $f'(x) = -12\sin x\cos x = 0 \Rightarrow \sin x = 0$ or $\cos x = 0$

OR $\sin 2x = 0$, for $0 \le 2x \le 2\pi$

THEN

$x = 0, \frac{\pi}{2}, \pi$

(b) (i) translation in the *y*-direction of –1

(ii) 1.11 (1.10 from TRACE is subject to **AP**)

38. (a) **EITHER** $A \sin\left(\frac{\pi}{2}\right) + B = 3$ and $A \sin\left(\frac{3\pi}{2}\right) + B = -1$

$\Leftrightarrow A + B = 3, -A + B = -1$

$\Leftrightarrow A = 2, B = 1$

OR

Amplitude $= A = \frac{3-(-1)}{2} = \frac{4}{2} = 2$

Midpoint value $= B = \frac{3+(-1)}{2} = \frac{2}{2} = 1$

(b) $f(x) = 2\sin\left(\frac{\pi}{2}x\right) + 1$

$f'(x) = \left(\frac{\pi}{2}\right)2\cos\left(\frac{\pi}{2}x\right) + 0 = \pi\cos\left(\frac{\pi}{2}x\right)$

(c) (i) $y = k - \pi x$ is a tangent $\Rightarrow -\pi = \pi\cos\left(\frac{\pi}{2}x\right)$

$\Rightarrow -1 = \cos\left(\frac{\pi}{2}x\right)$

$\Rightarrow \frac{\pi}{2}x = \pi$ or 3π or ...

$\Rightarrow x = 2$ or 6 ...

Since $0 \le x \le 5$, we take $x = 2$, so the point is $(2, 1)$

(ii) Tangent line is: $y = -\pi(x-2) + 1$

$y = (2\pi + 1) - \pi x$

$k = 2\pi + 1$

(d) $f(x) = 2 \Rightarrow 2\sin\left(\frac{\pi}{2}x\right) + 1 = 2$

$\Rightarrow \sin\left(\frac{\pi}{2}x\right) = \frac{1}{2}$

$\Rightarrow \frac{\pi}{2}x = \frac{\pi}{6}$ or $\frac{5\pi}{6}$ or $\frac{13\pi}{6}$

$x = \frac{1}{3}$ or $\frac{5}{3}$ or $\frac{13}{3}$

39. (a) recognizing the amplitude is the radius: $a = \frac{8}{2} \Rightarrow a = 4$

(b) period = 30: $b = \frac{2\pi}{30} = \frac{\pi}{15}$

(c) recognizing $h'(t) = -0.5$

$-0.5 = \frac{4\pi}{15}\cos\left(\frac{\pi}{15}t\right) \Rightarrow t = 10.6, t = 19.4$

(d) $h(t) < 0$ so underwater; $h(t) > 0$ so not underwater

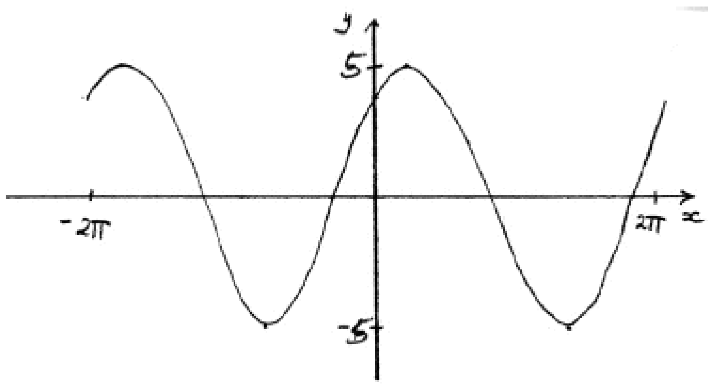
$h(19.4) = 4\sin\frac{19.4\pi}{15} + 2 = -1.19$

OR

solving $h(t) = 0$, graph showing region below x-axis, roots 17.5, 27.5

Hence, the bucket is underwater, yes

40. (a)



Note: *approximately sinusoidal shape,*
end points approximately correct, (–2π, 4), (2π, 4)
approximately correct position of graph,
(y-intercept (0, 4) maximum to right of y-axis).

(b) (i) 5 (ii) 2π (6.28) (iii) –0.927

(c) $f(x) = 5\sin(x + 0.927)$ (accept $p = 5$, $q = 1$, $r = 0.927$)

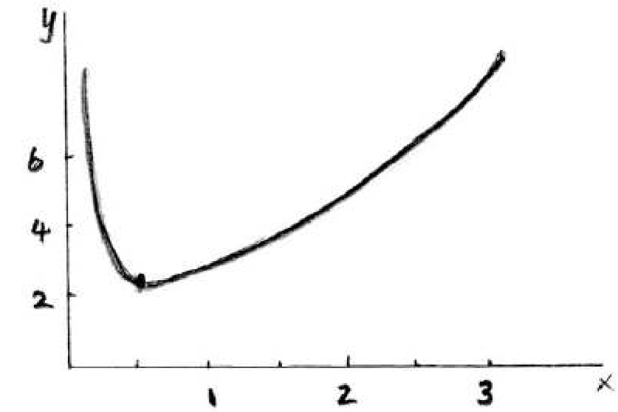
(d) (man or min)
one 3 s.f. value which rounds to one of –5.6, –2.5, 0.64, 3.8

(e) $k = -5$, $k = 5$

(f) $g'(x) = \frac{1}{x+1}$
$f'(x) = 3\cos x - 4\sin x$ $(5\cos(x + 0.927))$
$g'(x) = f'(x)$
$x = 0.511$

41.

(a) (i)



(ii) $g(x) = \frac{e^x}{\sqrt{x}}$

$$g'(x) = \frac{e^x\sqrt{x} - \frac{e^x}{2\sqrt{x}}}{x}$$

$$= \frac{(2x-1)e^x}{2x\sqrt{x}}$$

(iii) gradient is $-\frac{1}{g'(x)}$

$$= \frac{2x\sqrt{x}}{(1-2x)e^x}$$

(b) (i) $\frac{y-0}{x-1} = \frac{e^x}{\sqrt{x}(x-1)}$

(ii) **EITHER**

$$\frac{e^x}{\sqrt{x}(x-1)} = \frac{2x\sqrt{x}}{(1-2x)e^x}$$

$$x = 0.5454 \ldots$$

OR

$$D^2 = (x-1)^2 + y^2 = (x-1)^2 + \frac{e^{2x}}{x}$$

$$\frac{dD^2}{dx^2} = 2(x-1) + \frac{2e^{2x}x - e^{2x}}{x^2} = 0$$

$$x = 0.5454 \ldots$$

THEN

$$\text{distance} = \sqrt{(1-0.5454)^2 + \left(\frac{e^{0.5454}}{\sqrt{0.5454}}\right)^2}$$

$$= 2.38$$