EXERCISES [MAI 5.5]

OPTIMIZATION

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) $a = 2$, $b = 20$, $c = 9$, $d = 8$, $e = 32$

(b) $A = 12x - x^2$

(c) $\frac{dA}{dx} = 12 - 2x$

A is maximum when $12 - 2x = 0 \Rightarrow x =$

$\Rightarrow$ length = 6m and width = 6m

2. (a) $2x + 2y = 40 \Leftrightarrow x + y = 20 \Leftrightarrow y = 20 - x$

(b) $A = x(20 - x) = 20x - x^2$

(c) $\frac{dA}{dx} = 20 - 2x$

$20 - 2x = 0 \Leftrightarrow x = 10$

$\frac{d^2A}{dx^2} = -2 < 0$ so $x = 10$ gives a maximum.

It is a square of side $x = 10$ and the maximum area is $A = 100$

(d) The domain of $A = x(20 - x)$ is $0 \le x \le 20$.

At the endpoints $x = 0$ and $x = 20$, $A = 0$

3. (a) $xy = 100 \Leftrightarrow y = \frac{100}{x}$

$P = 2x + 2y = 2x + \frac{200}{x}$

(b) $\frac{dP}{dx} = 2 - \frac{200}{x^2}$

$2 - \frac{200}{x^2} = 0 \Leftrightarrow x^2 = 100 \Leftrightarrow x = 10$

$\frac{d^2P}{dx^2} = \frac{400}{x^3} > 0$ for $x = 10$, so it gives a minimum.

It is the square of side $x = 10$ and the minimum perimeter is $P = 40$

(c) The domain of $P = 2x + \frac{200}{x}$ is $x > 0$. When $x \to \infty$ then P can be as large as possible!

4. (a) **METHOD 1**

$l + 2w = 60 \Leftrightarrow l = 60 - 2w$

$A = w(60 - 2w) = 60w - 2w^2$

$\frac{\mathrm{d}A}{\mathrm{d}w} = 60 - 4w$

$60 - 4w = 0 \Leftrightarrow w = 15$

METHOD 2

$w + 2l = 60 \Leftrightarrow w = 60 - 2l$

$A = l(60 - 2l) = 60l - 2l^2$

$\frac{\mathrm{d}A}{\mathrm{d}l} = 60 - 4l$

$60 - 4l = 0 \Leftrightarrow l = 15$ and so $w = 30$

(b) $A_{\max} = 450$

5. let AB = x, AD = $\frac{525}{x}$

Cost C = 3(AD + BC + CD) + 11AB = $\frac{525}{x} \times 3 + \frac{525}{x} \times 3 + 11x + 3x = \frac{3150}{x} + 14x$

EITHER sketch of cost function,

min at $x = 15$, minimum cost is 420 (dollars)

OR using derivatives

$C'(x) = -\frac{3150}{x^2} + 14$

$\frac{-3150}{x^2} + 14 = 0 \Leftrightarrow x = 15$

minimum cost is $C = 420$ (dollars)

6. (a) (i) $l = 24 - 2x$ (ii) $w = 9 - 2x$

(b) $B = x(24 - 2x)(9 - 2x) = 4x^3 - 66x^2 + 216x$

(c) $\frac{\mathrm{d}B}{\mathrm{d}x} = 12x^2 - 132x + 216$

(d) (i) $\frac{\mathrm{d}B}{\mathrm{d}x} = 0 \Rightarrow x^2 - 11x + 18 = 0$

$\Rightarrow x = 2$ or $x = 9$ (not possible)

Therefore, $x = 2$ cm.

(ii) $B = 4(2)^3 - 66(2)^2 + 216(2)$ (or $2 \times 20 \times 5$) = 200 cm^3

7. (a) $x - 15$

(b) Profit = $(x - 15)(100\,000 - 4000x)$

$= 100000x - 4000x^2 - 1500\,000 + 60\,000x = 160\,000x - 4000x^2 - 1500\,000$

(c) (i) $\frac{\mathrm{d}P}{\mathrm{d}x} = 160000 - 8000x$

(ii) $160000 - 8000x = 0 \Leftrightarrow x = \frac{160000}{8000} \Leftrightarrow x = 20$

(d) Books sold = 100 000 – 4000 × 20 = 20000

8. (a) $D=\sqrt{(x-5)^2+(2x)^2}=\sqrt{x^2-10x+25+4x^2}=\sqrt{5x^2-10x+25}$

(b) $\dfrac{dS}{dx}=10x-10$

(c) $\dfrac{dS}{dx}=10x-10=0 \Leftrightarrow x=1$

[using table of signs or 2nd derivative test we easily see it gives a min]

(i) $S_{\min}=20$

(ii) $D_{\min}=\sqrt{20}$ ($\cong 4.47$)

(iii) P(1,2)

Notice :

We can also use the GDC graph for the function $D=\sqrt{5x^2-10x+25}$

It has a minimum at (1, 4.47)

Hence (i) The minimum distance is $D = 4.47$

(ii) The closest point is (1, 2)

9. **METHOD 1**

(a) $D=\sqrt{(a-3)^2+(a^2-0)^2}=\sqrt{a^2-6a+9+a^4}=\sqrt{a^4+a^2-6a+9}$

(b) (i) $\dfrac{dS}{da}=4a^3+2a-6$

(ii) $\left.\dfrac{dS}{da}\right|_{a=1}=4+2-6=0$

Either by a table of signs.

a	1	
$\dfrac{dS}{da}$	–	+

So minimum

OR by the 2nd derivative test

$S''=12a^2+2$, At $a=1$ $S''=14>0$ so minimum

(i) The point is $(1, 1^2)$ i.e. (1, 1)

(ii) The minimum distance is $D=\sqrt{5}$ ($\cong 2.24$)

Notice :

We can also use the GDC graph for the function $D=\sqrt{(a-3)^2+a^4}$

It has a minimum at (1, 2.236)

Hence (i) The point is $(1, 1^2)$ i.e. (1, 1)

(ii) The minimum distance is $D = 2.24$

10. (a) $f(x) = g(x) \Leftrightarrow x^2 = 4x - x^2 \Leftrightarrow 2x^2 - 4x = 0 \Leftrightarrow 2x(x-2) = 0 \Leftrightarrow x = 0$ or $x = 2$

Hence, $a = 2$

(b) $L = 4x - x^2 - x^2 = 4x - 2x^2$

(c) $\dfrac{dL}{dx} = 4 - 4x$

$4 - 4x = 0 \Leftrightarrow x = 1$

$\dfrac{d^2A}{dx^2} = -4 < 0$, so $x = 1$ gives a maximum.

$L_{\max} = 2$

11. (a) Base = $2a$, Height = $f(a) = 80 - a^4$

$S = 2a(80 - a^4) = 160a - 2a^5$

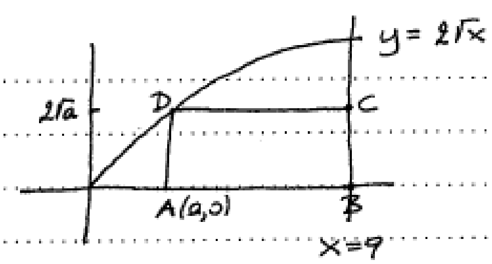
(c) $\dfrac{dS}{da} = 160 - 10a^4$

$160 - 10a^4 = 0 \Leftrightarrow a^4 = 16 \Leftrightarrow a = 2$

$\dfrac{d^2S}{da^2} = -40a^3$. For $a = 2$, $\dfrac{d^2S}{da^2} < 0$, hence max

$S_{\max} = 256$

12.



(a) length = $9-a$, width = $2\sqrt{a}$ $S = (9-a)2\sqrt{a}$

$\Rightarrow S = 18\sqrt{a} - 2a^{3/2}$

(b) $\dfrac{dS}{da} = \dfrac{18}{2\sqrt{a}} - 2\cdot\dfrac{3}{2}a^{1/2} = \dfrac{9}{\sqrt{a}} - 3\sqrt{a}$ $\left(= 9a^{-1/2} - 3\sqrt{a}\right)$

$\dfrac{dS}{da} = 0 \Leftrightarrow \dfrac{9}{\sqrt{a}} = 3\sqrt{a} \Leftrightarrow a = \dfrac{9}{3} \Leftrightarrow \boxed{a = 3}$

$\dfrac{d^2S}{da^2} = -\dfrac{9}{2}a^{-3/2} - \dfrac{3}{2\sqrt{a}} < 0$ for $a = 3$, hence $\boxed{\text{MAX}}$

Then $\boxed{S = 12\sqrt{3}}$

(c) $a = 0$ $a = 9$

B. Paper 2 questions (LONG)

13. (a) $h(2) = 24(2) - 2.4(2)^2 = 48 - 9.6 = 38.4$ cm

(b) (i) $\frac{dh}{dw} = 24 - 4.8w$

(ii) $24 - 4.8k = 7.2 \Leftrightarrow k = \frac{24 - 7.2}{4.8} = 3.5$ weeks

(iii) maximum height when $24 - 4.8w = 0 \Leftrightarrow w = \frac{24}{4.8} = 5$ weeks

height $= 24(5) - 2.4(5)^2 = 60$ cm

(c) 70 days = 10 weeks

$h(10) = 24(10) - 2.4(10)^2 = 0$

(height of zero indicates that the daffodil is lying on the ground)

14. (a) $2x + y$

(b) $2500 = 2x + y \Leftrightarrow 2500 - 2x = y$

(c) (i) Area $A(x) = xy = x(2500 - 2x) = 2500x - 2x^2$

(ii) $A'(x) = 2500 - 4x$

(iii) $A'(x) = 0 \Leftrightarrow 0 = 2500 - 4x \Leftrightarrow 4x = 2500 \Leftrightarrow x = 625$

(iv) $A(x) = 2500x - 2x^2$
$A(625) = 781250$ m^2

15. (a) $AE^2 + h^2 = 8^2 \Rightarrow AE = \sqrt{64 - h^2}$

(b) $V = \pi r^2(2h) = 2\pi h(AE^2) = 2\pi h(64 - h^2)$ cm^3

(c) (i) From (b) $V = 128\pi h - 2\pi h^3$

$\frac{dV}{dh} = 128\pi - 6\pi h^2 = 0 \Rightarrow h = \sqrt{\frac{64}{3}} = \pm 4.62$ cm (3 s.f.)

Test to show that V is maximum when $h = 4.62$ (either table or V'' test)

(ii) $AE^2 = 64 - h^2 = 64 - \frac{64}{3} = \frac{128}{3}$

$V_{max} = \pi r^2(2h) = \pi\left(\frac{128}{3}\right)\left(2\left(\sqrt{\frac{64}{3}}\right)\right) = 1238.22... = 1238$ cm^3 (nearest cm^3)

16. (a) $V = x^2h$

(b) $A = 2x^2 + 4xh$

(c) $1000 = x^2h \Leftrightarrow h = \frac{1000}{x^2}$

(d) $A = 2x^2 + 4x\left(\frac{1000}{x^2}\right) = 2x^2 + \frac{4000}{x} = 2x^2 + 4000x^{-1}$

(e) $\frac{dA}{dx} = 4x - 4000x^{-2}$

(f) $4x - 4000x^{-2} = 0 \Leftrightarrow 4x^3 = 4000 \Leftrightarrow x^3 = 1000 \Leftrightarrow x = 10$

(g) $A = 600$