EXERCISES [MAI 5.2-5.3]

BASIC DERIVATIVES – TANGENT AND NORMAL

SOLUTIONS

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A. Paper 1 questions (SHORT)

1.

Function	**Derivative**
$y = 7x^3 + 5x^2 + 2x + 3$	$\frac{dy}{dx} = 21x^2 + 10x + 2$
$y = \frac{7}{3}x^3 - \frac{5}{2}x^2 + \frac{1}{3}x + \frac{4}{5}$	$\frac{dy}{dx} = 7x^2 - 5x + \frac{1}{3}$
$y = \frac{7x^3}{3} - \frac{5x^2}{2} + \frac{x}{3} + \frac{4}{5}$	$\frac{dy}{dx} = 7x^2 - 5x + \frac{1}{3}$ (it's the same)
$y = 1 + \frac{2}{x} + \frac{3}{x^2} \quad (= 1 + 2x^{-1} + 3x^{-2})$	$\frac{dy}{dx} = -\frac{2}{x^2} - \frac{6}{x^3}$
$y = \frac{1}{3} + \frac{2}{5x} + \frac{3}{7x^2} \quad (= \frac{1}{3} + \frac{2}{5}x^{-1} + \frac{3}{7}x^{-2})$	$\frac{dy}{dx} = -\frac{2}{5x^2} + \frac{6}{7x^3}$
$y = x^2(1 + \frac{2}{x} + \frac{3}{x^2}) \quad (= x^2 + 2x + 3)$	$\frac{dy}{dx} = 2x + 2$
$y = \sqrt{x} + \sqrt[3]{x} \quad (= \sqrt{x} + x^{\frac{1}{3}})$	$\frac{dy}{dx} = \frac{1}{2\sqrt{x}} + \frac{1}{3}x^{-\frac{2}{3}} \quad (= \frac{1}{2\sqrt{x}} + \frac{1}{3\sqrt[3]{x^2}})$
$y = \sqrt{x^3} + \sqrt[3]{x^2} \quad (= x^{\frac{3}{2}} + x^{\frac{2}{3}})$	$\frac{dy}{dx} = \frac{3}{2}x^{\frac{1}{2}} + \frac{2}{3}x^{-\frac{1}{3}} \quad (= \frac{3\sqrt{x}}{2} + \frac{2}{3\sqrt[3]{x}})$
$y = \frac{1 + x + x^2}{x^2} \quad (= x^{-2} + x^{-1} + 1)$	$\frac{dy}{dx} = -2x^{-3} - x^{-2} \quad (= -\frac{2}{x^3} - \frac{1}{x^2})$
$y = \frac{3 + 5x + 7x^2}{2x^2} \quad (= \frac{3}{2}x^{-2} + \frac{5}{2}x^{-1} + \frac{7}{2})$	$\frac{dy}{dx} = -3x^{-3} - \frac{5}{2}x^{-2} \quad (= -\frac{3}{x^3} - \frac{5}{2x^2})$

2. (a) $f'(x) = 10x$

(b) $f'(1) = 10$

(c) $f'(x) = 20 \Leftrightarrow 10x = 20 \Leftrightarrow x = 2$, $y = 23$, thus (5,23)

3. (a) $f'(x) = \frac{2}{\sqrt{x}}$

(b) $f'(1) = 2$

(c) $f'(x) = 1 \Leftrightarrow \frac{2}{\sqrt{x}} = 1 \Leftrightarrow \sqrt{x} = 2 \Leftrightarrow x = 4$, $y = 8$, thus (4,8)

4. (a) $f'(x) = 3x^2 - 4x - 0 = 3x^2 - 4x$

(b) Gradient $= f'(2) = 3 \times 4 - 4 \times 2 = 4$

5. $f'(x) = 2x - 3b$

$f(1) = 0 \Leftrightarrow 1^2 - 3b + c + 2 = 0 \Leftrightarrow -3b + c + 3 = 0$

$f'(3) = 0 \Leftrightarrow 6 - 3b = 0 \Rightarrow 3b = 6 \Rightarrow b = 2$

$-6 + c + 3 = 0 \Rightarrow c = 3$

6. (a) At $x = 1$

(i) $y = f(1) + g(1) = 2 + 3 = 5$

(ii) $y = 2f(1) + 3g(1) = 4 + 9 = 13$

(iii) $y = f(1) + 5 = 2 + 5 = 7$

(b) (i) $\frac{dy}{dx} = f'(x) + g'(x)$. At $x = 1$, $\frac{dy}{dx} = f'(1) + g'(1) = 4 + 5 = 9$

(ii) $\frac{dy}{dx} = 2f'(x) + 3g'(x)$. At $x = 1$, $\frac{dy}{dx} = 2f'(1) + 3g'(1) = 8 + 15 = 23$

(iii) $\frac{dy}{dx} = f'(x) + 10x$. At $x = 1$, $\frac{dy}{dx} = f'(1) + 10 = 4 + 10 = 14$

7. (a) (i) 1 (ii) 0.5

(b) (i) 0 (ii) $-\frac{1}{2}$

8. (a) (i) 1
(ii) 2
(iii) $f'(14) = f'(2)\left(\text{or } f'(5) \text{ or } f'(8)\right) = -1$

(b) There are five repeated periods of the graph, each with two solutions,
ie number of solutions is $5 \times 2 = 10$

TANGENT LINE – NORMAL LINE

9. $f(x) = 2x^2 - 12x + 10$

(a) $f'(x) = 4x - 12$

(b) At $x = 2$, $y = 6$, Point (2,6)

$m_T = -4$, $m_N = \frac{1}{4}$

Tangent line: $y + 6 = -4(x - 2)$ (i.e. $y = -4x + 2$)

Normal line: $y + 6 = \frac{1}{4}(x - 2)$ (i.e. $y = \frac{1}{4}x - \frac{13}{2}$)

(a) At $x = 3$, $y = -8$, Point (3, –8)

$m_T = 0$

Tangent line: $y = -8$, Normal line: $x = 3$

10. $f(x) = 2x^2 - 12x + 10$

(a) $f'(x) = 4x - 12$

(b) (i) $m_1 = 4$

(ii) $4x - 12 = 4 \Leftrightarrow x = 4$, thus $y = -6$. Point (4,-6)

(c) (i) $m_2 = 8$

(ii) $4x - 12 = 8 \Leftrightarrow x = 5$, thus $y = 0$. Point (5,0)

(iii) $y - 0 = 8(x - 5) \Leftrightarrow y = 8x - 40$

11. $y = x^3 + 1 \quad \frac{dy}{dx} = 3x^2$

At $x = 1$, $m_T = 3$, $m_N = -\frac{1}{3}$

Equation of tangent: $y - 2 = 3(x - 1) \Rightarrow y = 3x - 1$

Equation of normal: $y - 2 = -\frac{1}{3}(x - 1)$ (**OR** $x + 3y - 7 = 0$ **OR** $y = -\frac{1}{3}x + 2\frac{1}{3}$)

12. $f'(x) = 12x^2 + 2$

When $x = 1, f(1) = 6$, Point (1,6)

When $x = 1, f'(1) = 14$, $m_T = 14$

Equation is $y - 6 = -\frac{1}{14}(x-1)$ $\left(y = -\frac{1}{14}x + \frac{85}{14}, y = -0.0714x + 6.07\right)$

13. $y = x^2 - x \quad \frac{dy}{dx} = 2x - 1.$

Line parallel to $y = 5x \Rightarrow 2x - 1 = 5 \Rightarrow x = 3$ so $y = 6$, Point (3, 6)

14. $f'(x) = 4kx^3$

$m_N = -\frac{1}{8}$, thus $m_T = 8$

$4kx^3 = 8 \Rightarrow kx^3 = 2$

substituting $x = 1$, $k = 2$

15. (a) $f'(x) = 6x - 5$

(b) $f'(p) = 7 \Rightarrow 6p - 5 = 7 \Rightarrow p = 2$

(c) Setting $y(2) = f(2)$

$k + 2 = 5 \Rightarrow k = 3$

16. (a) $f(1) = 3 \Rightarrow p + q = 3$

$f'(x) = 2px + q$

$f'(1) = 8 \Rightarrow 2p + q = 8$

$p = 5, q = -2$

17. (a) **METHOD 1 (directly by GDC)**

The equation of the tangent is $y = -4x - 8$.

METHOD 2

For $x = -1$, $y = -4$ and $\frac{dy}{dx} = 3x^2 + 8x + 1$, $m_T = -4$

Therefore, the tangent equation is $y + 4 = -4(x + 1) \Rightarrow y = -4x - 8$.

(b) This tangent meets the curve when $-4x - 8 = x^3 + 4x^2 + x - 6$

The required point of intersection is $(-2, 0)$.

18. For the curve, $y = 7$ when $x = 1 \Rightarrow a + b = 14$, and

$\frac{dy}{dx} = 6x^2 + 2ax + b = 16$ when $x = 1 \Rightarrow 2a + b = 10$.

Solving gives $a = -4$ and $b = 18$.

B. Paper 2 questions (LONG)

19. (a) (i) $p = 1, q = 5$ (or $p = 5, q = 1$)

(ii) $x = 3$ (must be an equation)

(b) $y = (x - 1)(x - 5) = x^2 - 6x + 5 = (x - 3)^2 - 4$ $(h = 3, k = -4)$

(c) $\frac{dy}{dx} = 2(x-3)$ $(= 2x - 6)$

(d) When $x = 0$, $\frac{dy}{dx} = -6$

$y - 5 = -6(x - 0)$ ($y = -6x + 5$ or equivalent)

20. (a) $h = 3$ $k = 2$

(b) $f(x) = -(x-3)^2 + 2 = -x^2 + 6x - 9 + 2 = -x^2 + 6x - 7$

(c) $f'(x) = -2x + 6$

(d) (i) tangent gradient $= -2$ gradient of $L = \frac{1}{2}$

$y - 1 = \frac{1}{2}(x - 4) \Rightarrow y = \frac{1}{2}x - 1$

(ii) $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Leftrightarrow 2x^2 - 11x + 12 = 0 \Leftrightarrow$ $x = 1.5$ or $x = 4$ so $x = 1.5$

(OR by GDC $-x^2 + 6x - 7 = \frac{1}{2}x - 1 \Rightarrow x = 1.5$

21. (a) (i) $f'(x) = -x + 2$

(ii) $f'(0) = 2$

(b) Gradient of tangent at y-intercept $= f'(0) = 2$

$\Rightarrow$ gradient of normal $= \frac{1}{2}$ $(= -0.5)$

Therefore, equation of the normal is $y - 2.5 = -0.5(x - 0) \Rightarrow y = -0.5x + 2.5$

(c) (i) $-0.5x^2 + 2x + 2.5 = -0.5x + 2.5 \Rightarrow x = 0$ or $x = 5$

(ii) Curve and normal intersect when $x = 0$ or $x = 5$

Other point is when $x = 5 \Rightarrow y = -0.5(5) + 2.5 = 0$ (so other point $(5, 0)$)

22. (a) (i) $p=-2 \quad q=4$ (or $p=4, q=-2$)

(ii) $y=a(x+2)(x-4) \Leftrightarrow 8=a(6+2)(6-4) \Leftrightarrow 8=16a \Leftrightarrow a=\frac{1}{2}$

(iii) $y=\frac{1}{2}(x+2)(x-4) \Rightarrow y=\frac{1}{2}(x^2-2x-8) \Rightarrow y=\frac{1}{2}x^2-x-4$

(b) $\frac{dy}{dx}=x-1$

$x-1=7 \Leftrightarrow x=8, y=20$ (P is (8, 20))

(c) (i) when $x = 4$, $m_T = 4 - 1 = 3 \Rightarrow m_N = -\frac{1}{3}$

$y-0=-\frac{1}{3}(x-4) \quad \left(y=-\frac{1}{3}x+\frac{4}{3}\right)$

(ii) $\frac{1}{2}x^2-x-4=-\frac{1}{3}x+\frac{4}{3} \Leftrightarrow x=-\frac{8}{3}$ or $x=4$

$x=-\frac{8}{3}$ (–2.67)

23. (a) $f'(x) = 3x^2 - 6x - 24$

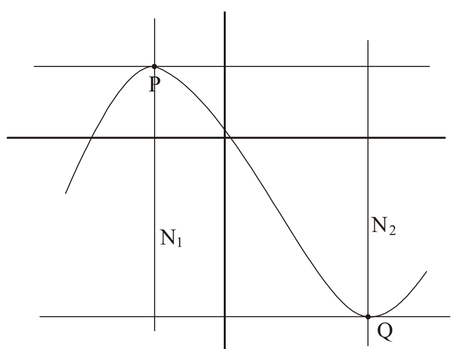
(b) Tangents parallel to the x-axis mean maximum and minimum (see graph)

EITHER by GDC P(–2, 29) and Q(4, –79)

OR $f'(x) = 0 \Leftrightarrow 3x^2 - 6x - 24 = 0 \Leftrightarrow x = -2$ or $x = 4$

Coordinates are P(–2, 29) and Q(4, –79)

(c)



(i) (4, 29) (ii) (–2, –79)

24. $f(x)=3x^2$

(a) $f'(x)=6x$, $f'(a)=6a$

(b) At $x=a$, $y=3a^2$, $m_T=6a$

Tangent line: $y-3a^2=6a(x-a) \Rightarrow y-3a^2=6ax-6a^2$

$\Rightarrow y=6ax-3a^2$

(c) $-3a^2=-3 \Leftrightarrow a^2=1 \Leftrightarrow a=\pm 1$

(d) For $a=1$, the line is $y=6x-3$

For $a=-1$, the line is $y=-6x-3$

25. $f(x) = 2x^3$

(a) $f'(x) = 6x^2$

At $x = 1$, $y = 2$, Point (1,2)

$m_T = 6$

Tangent line: $y - 2 = 6(x - 1)$ (i.e. $y = 6x - 4$)

(b) At $x = a$, $y = 2a^3$, Point $(a, 2a^3)$

$m_T = 6a^2$

Tangent line: $y - 2a^3 = 6a^2(x - a) \Rightarrow y - 2a^3 = 6a^2x - 6a^3$

$\Rightarrow y = 6a^2x - 4a^3$

(c) $-4a^3 = 4 \Leftrightarrow a^3 = -1 \Leftrightarrow a = -1$

Hence, the tangent line is $y = 6x + 4$