EXERCISES [MAI 3.9-3.11]

VECTORS – DOT PRODUCT

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a)

$\overrightarrow{AC} = \boldsymbol{b}$	**true**
$\overrightarrow{DE} = \boldsymbol{b}$	false
$\overrightarrow{ED} = \boldsymbol{a}$	false

$\overrightarrow{BC} = \boldsymbol{a}$	true
$\overrightarrow{BC} = \boldsymbol{b}$	false
$\lvert\overrightarrow{BC}\rvert = \lvert\boldsymbol{b}\rvert$	true

$\overrightarrow{BD} = 3\boldsymbol{a}$	true
$\overrightarrow{CD} = 2\boldsymbol{b}$	false
$\overrightarrow{CD} = 2\boldsymbol{a}$	true

(b)

$\overrightarrow{OC} = \boldsymbol{a} + \boldsymbol{b}$
$\overrightarrow{OD} = 3\boldsymbol{a} + \boldsymbol{b}$
$\overrightarrow{AD} = 2\boldsymbol{a} + \boldsymbol{b}$
$\overrightarrow{BA} = \boldsymbol{a} - \boldsymbol{b}$

$\overrightarrow{AB} = \boldsymbol{b} - \boldsymbol{a}$
$\overrightarrow{CE} = 2\boldsymbol{a} - \boldsymbol{b}$
$\overrightarrow{BE} = 3\boldsymbol{a} - \boldsymbol{b}$
$\overrightarrow{EC} = -2\boldsymbol{a} + \boldsymbol{b}$

2. (a) $\overrightarrow{CD} = \overrightarrow{OD} - \overrightarrow{OC}$

(b) $\overrightarrow{OA} = \frac{1}{2}\overrightarrow{CD} = \frac{1}{2}(\overrightarrow{OD} - \overrightarrow{OC})$

(c) $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \overrightarrow{OD} - \frac{1}{2}(\overrightarrow{OD} - \overrightarrow{OC}) = \frac{1}{2}\overrightarrow{OD} + \frac{1}{2}\overrightarrow{OC}$

3. (a) $\overrightarrow{OG} = 5\boldsymbol{i} + 5\boldsymbol{j} + 5\boldsymbol{k}$

(b) $\overrightarrow{BD} = -5\boldsymbol{i} + 5\boldsymbol{k}$

(c) $\overrightarrow{EB} = 5\boldsymbol{i} + 5\boldsymbol{j} - 5\boldsymbol{k}$

4.

magnitude of $\boldsymbol{a}$	$\lvert\boldsymbol{a}\rvert = \sqrt{5^2 + 12^2} = 13$
magnitude of $\boldsymbol{b}$	$\lvert\boldsymbol{b}\rvert = \sqrt{1^2 + 2^2} = \sqrt{5}$
unit vector corresponding to $\boldsymbol{a}$	$\hat{\boldsymbol{a}} = \frac{1}{13}\begin{pmatrix}5\\12\end{pmatrix} = \begin{pmatrix}5/13\\12/13\end{pmatrix}$
unit vector corresponding to $\boldsymbol{b}$	$\hat{\boldsymbol{b}} = \frac{1}{\sqrt{5}}\begin{pmatrix}1\\2\end{pmatrix} = \begin{pmatrix}1/\sqrt{5}\\2/\sqrt{5}\end{pmatrix}$
$\boldsymbol{a} + \boldsymbol{b}$	$\begin{pmatrix}6\\14\end{pmatrix}$
$\boldsymbol{a} - \boldsymbol{b}$	$\begin{pmatrix}4\\10\end{pmatrix}$
$2\boldsymbol{a}$	$\begin{pmatrix}10\\24\end{pmatrix}$
$\boldsymbol{a} - 2\boldsymbol{b}$	$\begin{pmatrix}3\\8\end{pmatrix}$
$\lvert\boldsymbol{a} - 2\boldsymbol{b}\rvert$	$\sqrt{73}$
a vector $\boldsymbol{c}$ parallel to $\boldsymbol{a}$ of magnitude 39	$\boldsymbol{c} = 3\boldsymbol{a} = \begin{pmatrix}15\\36\end{pmatrix}$

5.

magnitude of $\boldsymbol{a}$	$\lvert\boldsymbol{a}\rvert = \sqrt{2^2+1^2+2^2} = 3$
magnitude of $\boldsymbol{b}$	$\lvert\boldsymbol{b}\rvert = \sqrt{1^2+1^2+1^2} = \sqrt{3}$
unit vector corresponding to $\boldsymbol{a}$	$\hat{\boldsymbol{a}} = \frac{1}{3}\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}2/3\\1/3\\2/3\end{pmatrix}$
unit vector corresponding to $\boldsymbol{b}$	$\hat{\boldsymbol{b}} = \frac{1}{\sqrt{3}}\begin{pmatrix}1\\1\\1\end{pmatrix} = \begin{pmatrix}1/\sqrt{3}\\1/\sqrt{3}\\1/\sqrt{3}\end{pmatrix}$
$\boldsymbol{a}+\boldsymbol{b}$	$\begin{pmatrix}3\\2\\3\end{pmatrix}$
$\boldsymbol{a}+2\boldsymbol{b}$	$\begin{pmatrix}4\\3\\4\end{pmatrix}$
$\lvert\boldsymbol{a}+2\boldsymbol{b}\rvert$	$\sqrt{4^2+3^2+4^2} = \sqrt{41}$
a vector $\boldsymbol{c}$ parallel to $\boldsymbol{a}$ of magnitude 6	$\boldsymbol{c} = 2\boldsymbol{a} = 2\begin{pmatrix}2\\1\\2\end{pmatrix} = \begin{pmatrix}4\\2\\4\end{pmatrix}$

6. $\boldsymbol{u} = \begin{pmatrix}3\\5\end{pmatrix}$ $\boldsymbol{v} = \begin{pmatrix}1\\-2\end{pmatrix}$

Then $a\begin{pmatrix}4\\3\end{pmatrix} = \begin{pmatrix}8\\b-2\end{pmatrix} \Rightarrow$ $4a = 8$ and $3a = b-2$

Whence $a = 2$ $b = 8$

7. (a) $\vec{u} = -\vec{i} + 2\vec{j}$ $\vec{v} = 3\vec{i} + 5\vec{j}$ $\vec{u} + 2\vec{v} = 5\vec{i} + 12\vec{j}$

(b) $\lvert\vec{u} + 2\vec{v}\rvert = \sqrt{5^2+12^2} = 13$

Vector $\vec{w} = \frac{26}{13}(5\vec{i} + 12\vec{j}) = 10\vec{i} + 24\vec{j}$

8. (a) (i) AB = $\begin{pmatrix}3\\-1\\1\end{pmatrix}$ (ii) AC= $\begin{pmatrix}2\\6\\0\end{pmatrix}$ (iii) AB + AC= $\begin{pmatrix}5\\5\\1\end{pmatrix}$

(b) OC= $\begin{pmatrix}4\\7\\3\end{pmatrix}$

(c) (i) |AB|= $\sqrt{11}$ (ii) the same i.e. $\sqrt{11}$

(d) D(1,8,2)

9.

$\boldsymbol{a} \cdot \boldsymbol{b} = 0$
$\boldsymbol{a}^2 = (5)(5) = 25$
$\boldsymbol{b}^2 = (5)(5) = 25$
$\overrightarrow{OE} \cdot \overrightarrow{AC} = 0$
$\overrightarrow{OA} \cdot \overrightarrow{AE} = (5)(10)(\cos 0°) = 50$
$\overrightarrow{OA} \cdot \overrightarrow{OC} = (5)(5\sqrt{2})(\cos 45°) = 25$

10.

$\boldsymbol{a} \cdot \boldsymbol{b}$	$= 5 + 24 = 29$
$\boldsymbol{a}^2$	$= 25 + 144 = 169$
$\boldsymbol{b}^2$	$= 1 + 4 = 5$
cosine of angle θ between $\boldsymbol{a}$ and $\boldsymbol{b}$	$\cos\theta = \frac{29}{13\sqrt{5}}$

11. (a) $\boldsymbol{c} \cdot \boldsymbol{d} = 3 \times 5 + 4 \times (-12) = -33$

(b) $\boldsymbol{c} + \boldsymbol{d} = \begin{pmatrix} 8 \\ -8 \end{pmatrix}$

(c) $|\boldsymbol{c}| + |\boldsymbol{d}| = 5 + 13 = 18$

12. $\boldsymbol{a} \cdot \boldsymbol{b} = \begin{pmatrix} 3 \\ 4 \end{pmatrix} \bullet \begin{pmatrix} -2 \\ 1 \end{pmatrix} = (3 \times -2) + (4 \times 1) = -2$

magnitudes $\left|\begin{pmatrix} 3 \\ 4 \end{pmatrix}\right| = \sqrt{25}\,(=5), \left|\begin{pmatrix} -2 \\ 1 \end{pmatrix}\right| = \sqrt{5}$

$\cos\theta = \frac{-2}{\sqrt{125}}$

13. $\begin{pmatrix} 1 \\ 2 \end{pmatrix} . \begin{pmatrix} 6 \\ -8 \end{pmatrix} = 6 - 16 = -10$

$\left|\begin{pmatrix} 1 \\ 2 \end{pmatrix}\right| = \sqrt{1^2 + 2^2} = \sqrt{5}, \left|\begin{pmatrix} 6 \\ -8 \end{pmatrix}\right| = \sqrt{6^2 + 8^2} = \sqrt{100} = 10$

$\cos\theta = \frac{-10}{10\sqrt{5}} = -\frac{1}{\sqrt{5}}$

$\theta \approx 117°$

14. $\cos\theta = \frac{\mathbf{a.b}}{|\mathbf{a}||\mathbf{b}|} = \frac{-4 + 14}{\sqrt{20}\sqrt{50}} = \frac{10}{10\sqrt{10}} = \frac{1}{\sqrt{10}}$ (= 0.3162)

$\theta = 72°$ (to the nearest degree)

15. (a) $\begin{pmatrix}60\\25\end{pmatrix}\bullet\begin{pmatrix}-30\\40\end{pmatrix}$ = 60 × (–30) + 25 × 40= –800

(b) (i) $d = \sqrt{90^2 + 15^2} = \sqrt{8325} = 15\sqrt{37}$ (≅ 91.2)

(ii) $\cos\theta = \dfrac{-800}{\sqrt{60^2+25^2}\sqrt{(-30)^2+40^2}}$ = –0.246...

θ = 104.25...° (or 255.75...°)
She turns through 104° (or 256°)

Note: Accept answers in radians ie 1.82 or 4.46.

16. $\boldsymbol{v}\cdot w = 2 + 3 + 2 = 7$
$|\boldsymbol{v}| = \sqrt{6}$ $|\boldsymbol{w}| = \sqrt{14}$

$\cos\theta = \dfrac{\boldsymbol{v}\cdot\boldsymbol{w}}{|\boldsymbol{v}||\boldsymbol{w}|} = \dfrac{7}{\sqrt{6}\sqrt{14}} \Rightarrow \theta = 0.702$ radians.

17. (a) $\cos \hat{POQ} = \dfrac{\vec{OP}\cdot\vec{OQ}}{|\vec{OP}||\vec{OQ}|} = \dfrac{-6}{\sqrt{14}\sqrt{24}}\left(= -\dfrac{6}{\sqrt{336}}\right)$

$\hat{POQ} = 109^\circ$ (1.90 radians)

(b) **METHOD 1**

area $\Delta POQ = \dfrac{1}{2}|\vec{PO}||\vec{OQ}|\sin \hat{POQ} = 8.66$

METHOD 2 (using the cross product)

area $\Delta POQ = \dfrac{1}{2}|\vec{OP}\times\vec{OQ}| = \dfrac{1}{2}|10\boldsymbol{i}+10\boldsymbol{j}+10\boldsymbol{k}| = \sqrt{75}\ \left(=5\sqrt{3}\right)$

18. (a) $\overrightarrow{AB} = \begin{pmatrix}4\\3\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}-3\\1\end{pmatrix}$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = 4(-3) + 3(1) = -9$

(b) $|\overrightarrow{AB}| = 5$ $|\overrightarrow{AC}| = \sqrt{10}$

cos BAC = $\dfrac{-9}{5\sqrt{10}}$ = –0.569 (3 s.f)

19. (a) $\overrightarrow{AB} = \begin{pmatrix}4\\3\\-1\end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix}-3\\1\\1\end{pmatrix}$

$\overrightarrow{AB}\cdot\overrightarrow{AC} = 4(-3) + 3(1) - 1(1) = -10$

(b) $|\overrightarrow{AB}| = \sqrt{26}$ $|\overrightarrow{AC}| = \sqrt{11}$

cos $\hat{BAC}$ = $\dfrac{-10}{\sqrt{26}\sqrt{11}}$ = –0.591 (to 3 s.f)

20. (a) (i) $\overrightarrow{AB} = \begin{pmatrix}-2\\-3\end{pmatrix}$ (ii) $\overrightarrow{AC} = \begin{pmatrix}3\\-2\end{pmatrix}$

(b) (i) $\overrightarrow{AB}\bullet\overrightarrow{AC} = (-2)(3) + (-3)(-2) = 0$

(ii) scalar product 0 ⇒ perpendicular, $\theta = 90°$

21. (a) AB = $\begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}$ and AC= $\begin{pmatrix} 2 \\ 6 \\ 0 \end{pmatrix}$, $\overrightarrow{AB} \cdot \overrightarrow{AC} = 0$

(b) BAC = 90º

(c) |AB|= $\sqrt{11}$ |AC|= $\sqrt{40} = 2\sqrt{10}$

area of ABC $\sqrt{110}$

22. (a) $\overrightarrow{OB} = \begin{pmatrix} 10 \\ 5 \end{pmatrix}$ $\overrightarrow{AC} = \begin{pmatrix} -3 \\ 6 \end{pmatrix}$

(b) $\overrightarrow{OB} \cdot \overrightarrow{AC} = (10 \times (-3)) + (5 \times 6) = 0$ hence Angle = 90°

23. (a) $\begin{pmatrix} 2x \\ x-3 \end{pmatrix} \bullet \begin{pmatrix} x+1 \\ 5 \end{pmatrix} = 0 \Rightarrow 2x(x+1) + (x-3)(5) = 0 \Rightarrow 2x^2 + 7x - 15 = 0$

(b) $x = \frac{3}{2}$ or $x = -5$

24. (a) $\begin{pmatrix} 6-3a \\ -10 \\ 2b-24 \end{pmatrix} = \begin{pmatrix} 3 \\ -10 \\ -4 \end{pmatrix} \Rightarrow a = 1,\ b = 10$

(b) $\boldsymbol{u}$ is a multiple of $\boldsymbol{v}$ $\frac{a}{3} = \frac{4}{1} = \frac{8}{b} \Rightarrow a = 12,\ b = 2$

(c) $\boldsymbol{u} \cdot \boldsymbol{v} = 0 \Rightarrow 3a + 4 + 8b = 0$

$|\boldsymbol{u}| = \sqrt{9+1+b^2} = \sqrt{14} \Rightarrow b^2 = 4$

If $b = 2$, $a = -20/3$ [rejected]

If $b = -2$, $a = 4$

Hence $a = 4$ and $b = -2$

25. (a) $\boldsymbol{u} \bullet \boldsymbol{v} = 8 + 3 + p$

$8 + 3 + p = 0 \Rightarrow p = -11$

(b) $|\boldsymbol{u}| = \sqrt{2^2 + 3^2 + (-1)^2} = \sqrt{14}$, 3.74

$q\sqrt{14} = 14 \Rightarrow q = \sqrt{14}$ $(=3.74)$

26. $\boldsymbol{v} = \begin{pmatrix} 3 \\ 4 \\ 1 \end{pmatrix}$, $\boldsymbol{w} = \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix}$ $\boldsymbol{v} + p\boldsymbol{w} = \begin{pmatrix} 3+p \\ 4+2p \\ 1-3p \end{pmatrix}$

$\begin{pmatrix} 3+p \\ 4+2p \\ 1-3p \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ -3 \end{pmatrix} = 0 \Rightarrow 1(3+p) + 2(4+2p) - 3(1-3p) = 0$

$3 + p + 8 + 4p - 3 + 9p = 0 \Rightarrow 14p + 8 = 0 \Rightarrow p = \left(-\frac{8}{14}\right)$

27. $2\boldsymbol{a}-\boldsymbol{b}=2\begin{pmatrix}3\\2\\-1\end{pmatrix}-\begin{pmatrix}1\\5\\2\end{pmatrix}=\begin{pmatrix}5\\-1\\-4\end{pmatrix}$

$\begin{pmatrix}5\\-1\\-4\end{pmatrix}\begin{pmatrix}2\\\boldsymbol{y}\\3\end{pmatrix}=0\Rightarrow 10-y\ -12=0\Rightarrow y=-2$

28. $\boldsymbol{a}\bullet\boldsymbol{b}=3$

$\boldsymbol{s}=3\begin{pmatrix}3\\1\\\lambda\end{pmatrix}+\begin{pmatrix}\mu\\-2\\1\end{pmatrix}=\begin{pmatrix}9+\mu\\1\\3\lambda+1\end{pmatrix}$

$\boldsymbol{s}\bullet\boldsymbol{\alpha}=0\Rightarrow\begin{pmatrix}9+\mu\\1\\3\lambda+1\end{pmatrix}\bullet\begin{pmatrix}2\\3\\-1\end{pmatrix}=0$

$18+2\mu+3-3\lambda-1=0\Leftrightarrow 20+2\mu=3\lambda\Leftrightarrow\lambda=\dfrac{20+2\mu}{3}$

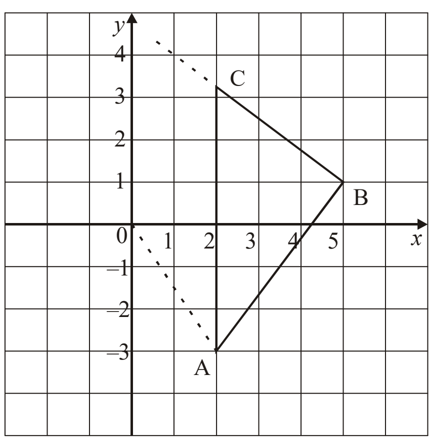
29. $\boldsymbol{a}\bullet\boldsymbol{b}=\begin{pmatrix}1\\-2\\3\end{pmatrix}\bullet\begin{pmatrix}3\\-2\\m\end{pmatrix}=7+3m$

$|\boldsymbol{a}|=\sqrt{14}\quad|\boldsymbol{b}|=\sqrt{13+m^2}$

$\cos 30^\circ=\dfrac{7+3m}{\sqrt{14}\sqrt{13+m^2}}$

$m=2.27,\ m=25.7$

30. (a)



(b) $\overrightarrow{OC}=\begin{pmatrix}2\\3.25\end{pmatrix}$

31. $\boldsymbol{a} = \begin{pmatrix} 12 \\ 5 \end{pmatrix}$, $\boldsymbol{b} = \begin{pmatrix} 6 \\ 8 \end{pmatrix}$

(a) $|\boldsymbol{a}| = \sqrt{12^2 + 5^2} = 13$

(b) $|\boldsymbol{b}| = \sqrt{6^2 + 8^2} = 10$

=> unit vector in direction of $\boldsymbol{b} = \dfrac{1}{10}\begin{pmatrix} 6 \\ 8 \end{pmatrix} = \begin{pmatrix} 0.6 \\ 0.8 \end{pmatrix}$

(c) $\cos\theta = \dfrac{12(6) + 5(8)}{13(10)} = \dfrac{112}{130} = \dfrac{56}{65}$

(d) **METHOD A**

$\cos\theta = \dfrac{|OC|}{|OA|} = \dfrac{|OC|}{|\boldsymbol{a}|} \Rightarrow |OC| = |\boldsymbol{a}|\cos\theta$

Hence $\boldsymbol{a} \cdot \boldsymbol{b} = |\boldsymbol{a}||\boldsymbol{b}|\cos\theta = |\boldsymbol{b}||OC|$

METHOD B

$\cos\theta = \dfrac{|OC|}{|OA|} = \dfrac{|OC|}{13} \Rightarrow \dfrac{56}{65} = \dfrac{|OC|}{13} \quad \Rightarrow |OC| = 11.2$

LHS = $\boldsymbol{a} \cdot \boldsymbol{b} = 12 \times 6 + 5 \times 8 = 112$

RHS = $|\boldsymbol{b}||OC| = 10 \times 11.2 = 112$

B. Paper 2 questions (*LONG*)

32. (a) (i) $\vec{AB} = \begin{pmatrix} 2 \\ -4 \\ -2 \end{pmatrix}$ (ii) $\vec{AD} = \begin{pmatrix} 2 \\ k-5 \\ -2 \end{pmatrix}$

(b) scalar product = 0 $\Rightarrow 4 - 4(k - 5) + 4 = 0 \Rightarrow -4k + 28 = 0 \Rightarrow k = 7$

(c) $\vec{AD} = \begin{pmatrix} 2 \\ 2 \\ -2 \end{pmatrix}$ $\vec{BC} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$

$\vec{OC} - \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix} \Rightarrow \vec{OC} = \begin{pmatrix} 4 \\ 2 \\ 1 \end{pmatrix}$

(d) **METHOD 1**

$\vec{BA} \cdot \vec{BC} = -2(1) + 4(1) + 2(-1) = 0$

$\cos A\hat{B}C = 0$

METHOD 2

$\vec{BC}$ parallel to $\vec{AD}$ (may show this on a diagram with points labelled)

$\vec{BC} \perp \vec{AB}$ (may show this on a diagram with points labelled)

$A\hat{B}C = 90°$

$\cos A\hat{B}C = 0$

33. (a) (i) $\overrightarrow{AB} = \begin{pmatrix} 6 \\ 4 \\ 4 \end{pmatrix} - \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix} \Rightarrow \overrightarrow{AB} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix}$ (ii) $\overrightarrow{AD} = \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix}$

(iii) $\overrightarrow{AC} = \overrightarrow{AB} + \overrightarrow{AD} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} + \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 6 \\ 5 \\ 3 \end{pmatrix}$

(b) There are several ways: *e.g.* $\overrightarrow{OC} = \overrightarrow{OA} + \overrightarrow{AC}$, $\overrightarrow{OC} = \overrightarrow{OB} + \overrightarrow{AD}$, or using AB = DC

coordinates of C are (7, 7, 6)

(c) (i) $\overrightarrow{AB} \bullet \overrightarrow{AD} = 5(1) + 2(3) + 1(2) = 13$

(ii) $\cos A = \dfrac{\overrightarrow{AB} \bullet \overrightarrow{AD}}{|\overrightarrow{AB}||\overrightarrow{AD}|} = \dfrac{13}{20.493...} \Rightarrow \hat{A} = 0.884\ (50.6°)$

(d) **METHOD 1**

area $= 2\left(\frac{1}{2}|\overrightarrow{AD}||\overrightarrow{AB}| \sin D\hat{A}B\right)$ area $= 2\left(\frac{1}{2}(3.741...)(5.477...) \sin 0.883...\right)$

area = 15.8

METHOD 2

Using cross product $\vec{AB} \times \vec{AD} = \begin{pmatrix} 5 \\ 2 \\ 1 \end{pmatrix} \times \begin{pmatrix} 1 \\ 3 \\ 2 \end{pmatrix} = \begin{pmatrix} 1 \\ -9 \\ 13 \end{pmatrix}$,

Area $= |\vec{AB} \times \vec{AD}| = \sqrt{1^2 + 9^2 + 13^2} = \sqrt{251} \cong 15.8$

34. (a) $\overrightarrow{OR} = \overrightarrow{PQ} = \begin{pmatrix} 10 \\ 1 \end{pmatrix} - \begin{pmatrix} 7 \\ 3 \end{pmatrix} = \begin{pmatrix} 3 \\ -2 \end{pmatrix}$

(b) $\left|\overrightarrow{PO}\right| = \sqrt{(-7)^2 + (-3)^2} = \sqrt{58}$, $\left|\overrightarrow{PQ}\right| = \sqrt{3^2 + (-2)^2} = \sqrt{13}$

$\overrightarrow{PO} \cdot \overrightarrow{PQ} = -21 + 6 = -15$

$\cos\ O\hat{P}Q = \dfrac{-15}{\sqrt{58}\sqrt{13}} = \dfrac{-15}{\sqrt{754}}$

(c) (i) Since $O\hat{P}Q + P\hat{Q}R = 180°$

(ii) $\sin\ P\hat{Q}R = \sqrt{1 - \left(\dfrac{15}{\sqrt{754}}\right)^2} = \sqrt{\dfrac{529}{754}} = \dfrac{23}{\sqrt{754}}$

(iii) Area of OPQR = 2 (area of triangle PQR)

$= 2 \times \dfrac{1}{2}\left|\overrightarrow{PQ}\right| \times \left|\overrightarrow{QR}\right| \times \sin P\hat{Q}R$

$= 2 \times \dfrac{1}{2}\sqrt{13}\sqrt{58}\dfrac{23}{\sqrt{754}}$ = 23 sq units.

35. (a) (i) $\overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA} = \begin{pmatrix} -3 \\ -1 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} -5 \\ 1 \end{pmatrix}$

(ii) $\left|\overrightarrow{AB}\right| = \sqrt{25+1} = \sqrt{26}$ (= 5.10 to 3 sf)

(b) $\overrightarrow{AD} = \overrightarrow{OD} - \overrightarrow{OA} = \begin{pmatrix} d \\ 23 \end{pmatrix} - \begin{pmatrix} 2 \\ -2 \end{pmatrix} = \begin{pmatrix} d-2 \\ 25 \end{pmatrix}$

(c) (i) $B\hat{A}D = 90° \Rightarrow \overrightarrow{AB} \bullet \overrightarrow{AD} = 0 \Rightarrow \begin{pmatrix} -5 \\ 1 \end{pmatrix} \bullet \begin{pmatrix} d-2 \\ 25 \end{pmatrix} = 0$

$-5d + 10 + 25 = 0 \Rightarrow d = 7$

(ii) $\overrightarrow{OD} = \begin{pmatrix} 7 \\ 23 \end{pmatrix}$

(d) $\overrightarrow{AD} = \overrightarrow{BC}$

$\overrightarrow{BC} = \begin{pmatrix} 5 \\ 25 \end{pmatrix} \Rightarrow \overrightarrow{OC} = \begin{pmatrix} 2 \\ 24 \end{pmatrix}$

(e) $\left|\overrightarrow{AD}\right|\left(\text{or} \left|\overrightarrow{BC}\right|\right) = \sqrt{5^2 + 25^2} = \sqrt{650}$

Area = $\sqrt{26} \times \sqrt{650}$ = 130

36. (a) $\left|\overrightarrow{OA}\right| = 6$, $\left|\overrightarrow{OB}\right| = 6$, $\left|\overrightarrow{OC}\right| = \left|\begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix}\right| = \sqrt{25+11} = 6 \Rightarrow$

A, B, C are on the circle.

(b) $\overrightarrow{AC} = \overrightarrow{OC} - \overrightarrow{OA} = \begin{pmatrix} 5 \\ \sqrt{11} \end{pmatrix} - \begin{pmatrix} 6 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}$

(c) $\cos O\hat{A}C = \dfrac{\overrightarrow{AO} \cdot \overrightarrow{AC}}{\left|AO\right|\left|AC\right|} = \dfrac{\begin{pmatrix} -6 \\ 0 \end{pmatrix} . \begin{pmatrix} -1 \\ \sqrt{11} \end{pmatrix}}{6\sqrt{1+11}} = \dfrac{6}{6\sqrt{12}} = \dfrac{1}{2\sqrt{3}} = \dfrac{\sqrt{3}}{6}$

(d) A number of possible methods here. The quickest method is:

ΔABC has base AB = 12 and height = $\sqrt{11}$

$\Rightarrow$ area = $\dfrac{1}{2} \times 12 \times \sqrt{11} = 6\sqrt{11}$

37. (a) $|\boldsymbol{a}| = |\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}|^2 = |\boldsymbol{b}|^2$

$\Leftrightarrow \boldsymbol{a}^2 = \boldsymbol{b}^2$

$\Leftrightarrow \boldsymbol{a}^2 - \boldsymbol{b}^2 = 0$

$\Leftrightarrow (\boldsymbol{a}+\boldsymbol{b})\cdot(\boldsymbol{a}-\boldsymbol{b}) = 0$

$\Leftrightarrow \boldsymbol{a} + \boldsymbol{b} \perp \boldsymbol{a} - \boldsymbol{b}$

(b) $|\boldsymbol{a}+\boldsymbol{b}| = |\boldsymbol{a}-\boldsymbol{b}| \Leftrightarrow |\boldsymbol{a}+\boldsymbol{b}|^2 = |\boldsymbol{a}-\boldsymbol{b}|^2$

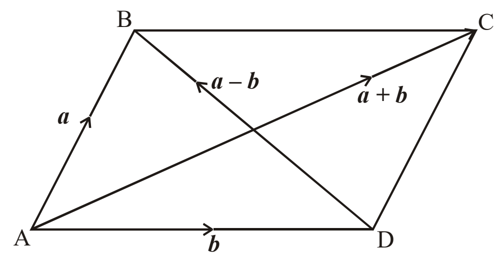
$\Leftrightarrow (\boldsymbol{a}+\boldsymbol{b})^2 = (\boldsymbol{a}-\boldsymbol{b})^2$

$\Leftrightarrow \boldsymbol{a}^2 + 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}^2 = \boldsymbol{a}^2 - 2\boldsymbol{a}\cdot\boldsymbol{b} + \boldsymbol{b}^2$

$\Leftrightarrow 4\boldsymbol{a}\cdot\boldsymbol{b} = 0 \Leftrightarrow \boldsymbol{a}\cdot\boldsymbol{b} = 0$

$\Leftrightarrow \boldsymbol{a} \perp \boldsymbol{b}$

(c)



(i) a parallelogram is a rhombus if and only if the diagonals are equal

(ii) a parallelogram is a rectangle if and only if the diagonals bisect each other

(iii) a parallelogram is a square if and only if the diagonals are equal and bisect each other

38. (a) $AH = 5$ cm, $HC = 3\sqrt{5}$ cm, $AC = 2\sqrt{13}$ cm

$$\cos A\hat{H}C = \frac{AH^2 + CH^2 - AC^2}{2(AH)(CH)} = \frac{25+45-52}{30\sqrt{5}}$$

i.e. $A\hat{H}C = 74.4°$ (to the nearest one-tenth of a degree)

(b) A(0,0,0), H(0,4,3), C(6,4,0)

$$\overrightarrow{HA} = \begin{pmatrix} 0 \\ -4 \\ -3 \end{pmatrix}, \overrightarrow{HC} = \begin{pmatrix} 6 \\ 0 \\ -3 \end{pmatrix}$$

$$\cos A\hat{H}C = \frac{9}{5\sqrt{45}} \Rightarrow A\hat{H}C = 74.4°$$

39. (a) $\overrightarrow{\text{AB}} = \boldsymbol{b} - \boldsymbol{a}$, $\overrightarrow{\text{BC}} = \boldsymbol{c} - \boldsymbol{b}$, $\overrightarrow{\text{CA}} = \boldsymbol{a} - \boldsymbol{c}$

(b) $\overrightarrow{\text{OA}} \perp \overrightarrow{\text{BC}} \Rightarrow \boldsymbol{a} \bullet (\boldsymbol{c} - \boldsymbol{b}) = 0 \Rightarrow \boldsymbol{a} \bullet \boldsymbol{c} - \boldsymbol{a} \bullet \boldsymbol{b} = 0 \Rightarrow \boldsymbol{a} \bullet \boldsymbol{c} = \boldsymbol{a} \bullet \boldsymbol{b}$

$\overrightarrow{\text{OB}} \perp \overrightarrow{\text{CA}} \Rightarrow \boldsymbol{b} \bullet (\boldsymbol{a} - \boldsymbol{c}) = 0 \Rightarrow \boldsymbol{b} \bullet \boldsymbol{a} - \boldsymbol{b} \bullet \boldsymbol{c} = 0 \Rightarrow \boldsymbol{b} \bullet \boldsymbol{a} = \boldsymbol{b} \bullet \boldsymbol{c}$

Therefore,

$\boldsymbol{b} \bullet \boldsymbol{c} = \boldsymbol{a} \bullet \boldsymbol{c} \Rightarrow \boldsymbol{b} \bullet \boldsymbol{c} - \boldsymbol{a} \bullet \boldsymbol{c} = 0 \Rightarrow (\boldsymbol{b} - \boldsymbol{a}) \bullet \boldsymbol{c} = 0$

$\Rightarrow \overrightarrow{\text{OC}}$ is perpendicular to $\overrightarrow{\text{AB}}$