EXERCISES [MAI 3.12-3.13]

EQUATIONS OF LINES

SOLUTIONS

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A. Paper 1 questions (SHORT)

1.

Passing through	Parallel to	Equation of line
A(3,5)	$\vec{b}=\begin{pmatrix}1\\7\end{pmatrix}$	$\vec{r}=\begin{pmatrix}3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\7\end{pmatrix}$
the origin	$\vec{b}=\begin{pmatrix}2\\3\end{pmatrix}$	$\vec{r}=\begin{pmatrix}0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\3\end{pmatrix}$
A(3,5)	x-axis	$\vec{r}=\begin{pmatrix}3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\0\end{pmatrix}$
A(3,5)	y-axis	$\vec{r}=\begin{pmatrix}3\\5\end{pmatrix}+\lambda\begin{pmatrix}0\\1\end{pmatrix}$
A(1,3,5)	$\vec{b}=\begin{pmatrix}1\\7\\2\end{pmatrix}$	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\7\\2\end{pmatrix}$
the origin	$\vec{b}=\begin{pmatrix}1\\4\\3\end{pmatrix}$	$\vec{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}1\\4\\3\end{pmatrix}$
A(1,3,5)	x-axis	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$
A(1,3,5)	y-axis	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}0\\1\\0\end{pmatrix}$
A(1,3,5)	z-axis	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$

2.

Passing through	Equation of line
$A(3,5)$ and $B(4,12)$	$\vec{r}=\begin{pmatrix}3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\7\end{pmatrix}$
$A(1,-4)$ and $B(7,6)$	$\vec{r}=\begin{pmatrix}1\\-4\end{pmatrix}+\lambda\begin{pmatrix}6\\10\end{pmatrix}$
the origin and $B(7,6)$	$\vec{r}=\begin{pmatrix}0\\0\end{pmatrix}+\lambda\begin{pmatrix}7\\6\end{pmatrix}$

Passing through	Equation of line
$A(1,3,5)$ and $B(2,10,7)$	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}1\\7\\2\end{pmatrix}$
$A(1,3,5)$ and $B(0,5,3)$	$\vec{r}=\begin{pmatrix}1\\3\\5\end{pmatrix}+\lambda\begin{pmatrix}-1\\2\\-2\end{pmatrix}$
the origin and $B(2,10,7)$	$\vec{r}=\begin{pmatrix}0\\0\\0\end{pmatrix}+\lambda\begin{pmatrix}2\\10\\7\end{pmatrix}$

3. (i) $\boldsymbol{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}1\\0\\0\end{pmatrix}$ (ii) $\boldsymbol{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\1\\0\end{pmatrix}$ (iii) $\boldsymbol{r} = \begin{pmatrix}0\\0\\0\end{pmatrix} + \lambda\begin{pmatrix}0\\0\\1\end{pmatrix}$

4. (a) yes (b) no (c) no

5. Required vector will be parallel to $\begin{pmatrix}3\\-1\end{pmatrix} - \begin{pmatrix}-1\\4\end{pmatrix} = \begin{pmatrix}4\\-5\end{pmatrix}$

Hence required equation is $\boldsymbol{r} = \begin{pmatrix}-1\\4\end{pmatrix} + t\begin{pmatrix}4\\-5\end{pmatrix}$ (OR $\boldsymbol{r} = \begin{pmatrix}3\\-1\end{pmatrix} + s\begin{pmatrix}4\\-5\end{pmatrix}$).

6. (a) $\overrightarrow{\text{PQ}} = \begin{pmatrix}5\\-3\end{pmatrix}$

(b) Using $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$ $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\6\end{pmatrix} + t\begin{pmatrix}5\\-3\end{pmatrix}$

7. (a) Vector equation of a line $\boldsymbol{r} = \boldsymbol{a} + \lambda\boldsymbol{t}$, $\boldsymbol{a} = \begin{pmatrix}0\\0\end{pmatrix}$, $\boldsymbol{t} = \begin{pmatrix}2\\3\end{pmatrix} \Rightarrow \boldsymbol{r} = \lambda\begin{pmatrix}2\\3\end{pmatrix}$

(b) $y = \frac{3}{2}x$

8. Direction vector = $\begin{pmatrix}6\\5\end{pmatrix} - \begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}5\\2\end{pmatrix}$

$\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}1\\3\end{pmatrix} + t\begin{pmatrix}5\\2\end{pmatrix}$ **OR** $\begin{pmatrix}x\\y\end{pmatrix} = \begin{pmatrix}6\\5\end{pmatrix} + t\begin{pmatrix}5\\2\end{pmatrix}$

9. $x = 1 - 2t \quad y = 2 + 3t$

$\frac{x-1}{-2} = \frac{y-2}{3} \Leftrightarrow 3x + 2y = 7$

10. Perpendicular vector: either $\begin{pmatrix}3\\-2\end{pmatrix}$ or $\begin{pmatrix}-3\\2\end{pmatrix}$

$\vec{r} = \begin{pmatrix}4\\-1\end{pmatrix} + \lambda\begin{pmatrix}3\\-2\end{pmatrix} \Rightarrow \begin{cases} x = 4 + 3\lambda \\ y = -1 - 2\lambda \end{cases} \Rightarrow \frac{x-4}{3} = \frac{y+1}{-2}$

$2(x - 4) + 3(y + 1) = 0 \Rightarrow 2x - 8 + 3y + 3 = 0 \Rightarrow 2x + 3y = 5$

OR

Gradient of a line parallel to the vector $\begin{pmatrix}2\\3\end{pmatrix}$ is $\frac{3}{2}$

Gradient of a line perpendicular to this line is $-\frac{2}{3}$

So the equation is $y + 1 = -\frac{2}{3}(x - 4) \Rightarrow 3y + 3 = -2x + 8 \Rightarrow 2x + 3y = 5$

11. (a) $\overrightarrow{\text{AB}} = \overrightarrow{\text{OB}} - \overrightarrow{\text{OA}} \left(= \begin{pmatrix}1\\5\\3\end{pmatrix} - \begin{pmatrix}3\\2\\1\end{pmatrix} \right)$ $\overrightarrow{\text{AB}} = \begin{pmatrix}-2\\3\\2\end{pmatrix}$

(b) Using $\boldsymbol{r} = \boldsymbol{a} + t\boldsymbol{b}$: $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}3\\2\\1\end{pmatrix} + t\begin{pmatrix}-2\\3\\2\end{pmatrix}$ or $\begin{pmatrix}x\\y\\z\end{pmatrix} = \begin{pmatrix}1\\5\\3\end{pmatrix} + t\begin{pmatrix}-2\\3\\2\end{pmatrix}$

12. B, i.e $\boldsymbol{r} = \begin{pmatrix}4\\4\end{pmatrix} + t\begin{pmatrix}6\\2\end{pmatrix}$ and D, i.e $\boldsymbol{r} = \begin{pmatrix}7\\5\end{pmatrix} + t\begin{pmatrix}3\\1\end{pmatrix}$

13. (a) $AB = \begin{pmatrix}1\\1\\1\end{pmatrix}$ $BC = \begin{pmatrix}3\\3\\3\end{pmatrix}$, Hence $BC = 3AB$. Since $BC // AB$ and they have a common point A,B,C are collinear. (Notice: You can also consider $AC = 4AB$)

(b) $\boldsymbol{r} = \begin{pmatrix}1\\1\\1\end{pmatrix} + \lambda\begin{pmatrix}3\\4\\5\end{pmatrix}$

(c) $\cos BAD = \dfrac{AB \cdot AD}{|AB| \cdot |AD|} = \dfrac{12}{\sqrt{150}}$

14. Angle between lines = angle between direction vectors $\begin{pmatrix}4\\3\end{pmatrix}$ **and** $\begin{pmatrix}1\\-1\end{pmatrix}$.

$\cos\theta = \dfrac{1}{5\sqrt{2}} = 0.1414 \Rightarrow \theta = 81.9°$ (3 sf), (1.43 radians)

15. Direction vectors are $\boldsymbol{a} = \boldsymbol{i} - 3\boldsymbol{j}$ and $\boldsymbol{b} = \boldsymbol{i} - \boldsymbol{j}$.

$\cos\theta = \dfrac{\boldsymbol{a}\cdot\boldsymbol{b}}{|\boldsymbol{a}||\boldsymbol{b}|}\left(= \dfrac{4}{\sqrt{10}\sqrt{2}}\right) = \dfrac{4}{\sqrt{20}}$

16.

The direction vectors of L_1 and L_2 are

$L_1 = \begin{pmatrix}2\\3\\-1\end{pmatrix}, L_2 = \begin{pmatrix}-1\\4\\2\end{pmatrix}$

$L_1 \cdot L_2 = 8$

$|L_1| = \sqrt{14}$, $|L_2| = \sqrt{21}$

$\cos\theta = \dfrac{8}{\sqrt{14}\sqrt{21}}$

$\theta = 1.09$ radians $(62.2°)$

17. **METHOD 1**

At point of intersection:

$5 + 3\lambda = -2 + 4t$

$1 - 2\lambda = 2 + t$

Solve the linear system: $\lambda = -1$ (or $t = 1$)

$\overrightarrow{OP} = \begin{pmatrix}2\\3\end{pmatrix}$

METHOD 2 (changing to Cartesian coordinates)

$2x + 3y = 13$, $x - 4y = -10$

Solve the system

$\overrightarrow{OP} = \begin{pmatrix}2\\3\end{pmatrix}$

18.

(a) $l_1 \quad \boldsymbol{r} = \begin{pmatrix} 4+\lambda \\ 3+5\lambda \\ -2\lambda \end{pmatrix}$

for l_1 for $x = 2,\ \lambda = -2$
$\Rightarrow y = -7$
$\Rightarrow z = 4$
Therefore point fits on line.

(b) $4+\lambda = 2$ Eq(1)
$3+5\lambda = -1+2\mu$ Eq(2)
$-2\lambda = 3-3\mu$ Eq(3)

From Eq(1), $\lambda = -2$
From Eq(2), $3-10 = -1+2\mu$
$-7 = -1+2\mu$
$\mu = -3$
Substituting in Eq(3)
$\Rightarrow 4 = 3+9$
$\Rightarrow$ lines do not intersect

19. (a) the dot product of the direction vectors is 0. (b) (5,19).

(c) $7x - y = 16$ and $x + 7y = 138$ (d) x=5, y=19

20. (a) the first two equations give λ=2, μ=1 which satisfy the 3rd equation.

(b) (4,9,7)

(c) $\cos\theta = \dfrac{8}{\sqrt{21}\sqrt{13}} \Rightarrow \theta = 61^\circ$

21. $r_1 = r_2$
$2+s = 3-t,$
$5+2s = -3+3t,$
$3+3s = 8-4t$
solve the equations, finding **one** correct parameter ($s = -1$, $t = 2$)
the coordinates of T are (1, 3, 0)

22. Solve $1 + \lambda = 1 + 2\mu$, $1 + 2\lambda = 4 + \mu$, $1 + 3\lambda = 5 + 2\mu$
Solving, $\lambda = 2$, (or $\mu = 1$).

P has position vector $3\boldsymbol{i} + 5\boldsymbol{j} + 7\boldsymbol{k}$ or $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$

23. Line (AB) is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 4 \\ -1 \end{pmatrix} + \lambda \begin{pmatrix} 1 \\ 1 \\ -1 \end{pmatrix}$ Line (CD) is $\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 5 \\ 6 \\ 3 \end{pmatrix} + \mu \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix}$

at point of intersection of two lines
$1 + \lambda = 5 + 3\mu$
$4 + \lambda = 6 + 2\mu$
$-1 - \lambda = 3 + \mu$
solving simultaneously any two of these three equations gives
$\lambda = -2$ and $\mu = -2$ (only one value required). $\Rightarrow$ point of intersection $(-1, 2, 1)$
***Note**: Since question states that lines intersect, no need to check the 3rd equation*

24. The direction of the line is $\boldsymbol{v} = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $|\boldsymbol{v}| = 3$.

Therefore, the position vector of any point on the line 6 units from A is

$\boldsymbol{a} \pm 2\boldsymbol{v} = \begin{pmatrix} 3 \\ 0 \\ -2 \end{pmatrix} \pm 2\begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 7 \\ -4 \\ 0 \end{pmatrix}$ or $\begin{pmatrix} -1 \\ 4 \\ -4 \end{pmatrix}$. giving the point (7, –4, 0) or (–1, 4, –4).

OR

Any point on the line has the form P(3+2λ,-2λ,-2+λ).
The distance between A and B must be 6.
This will give two values for λ: λ=±2.
Thus the point is (7, –4, 0) or (–1, 4, –4).

25. (a) $\vec{AB} = \begin{pmatrix} 1 \\ -3 \end{pmatrix}$, $\vec{OR} = \begin{pmatrix} x \\ 3-3x \end{pmatrix}$

(b) $\vec{AB} \bullet \vec{OR} = x - 3(3-3x) = 0 \quad (10x - 9 = 0)$

R is $\left(\frac{9}{10}, \frac{3}{10}\right)$

26. (a) $\overrightarrow{AB} = \begin{pmatrix} 5 \\ -10 \\ 25 \end{pmatrix}$. Direction vector of line is $\begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix}$ (OR any multiple)

Therefore the equation of l in parametric form is
$x = \lambda + 1$,
$y = -2\lambda + 3$,
$z = 5\lambda - 17$
(or $x = \lambda + 6$, $y = -2\lambda - 7$, $z = 5\lambda + 8$, or any equivalent **parametric** form)

(b) P on l => P can be written as $(p + 1, -2p + 3, 5p - 17)$.

$\overrightarrow{OP} \perp l \Rightarrow \begin{pmatrix} p+1 \\ -2p+3 \\ 5p-17 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ -2 \\ 5 \end{pmatrix} = 0$

$p + 1 + 4p - 6 + 25p - 85 = 0 \Rightarrow 30p = 90 \Rightarrow p = 3$
Therefore P is (4, –3, –2)

27. (a) Only $B(8,3,9)$ lies on the line

(b) $d(A,P) = 6 \;\Rightarrow \lambda = 1 \text{ or } 5$ The points are (4,1,5) and (12,5,13)

(c) $d(O,P) = \sqrt{89} \;\Rightarrow \lambda = 2, \lambda = -\frac{38}{9}$ The points are (6,2,7) and $\left(-\frac{58}{9}, -\frac{38}{9}, -\frac{49}{9}\right)$

(d) $d(C,P) = \sqrt{54} \;\Rightarrow \lambda = 2, \lambda = -\frac{26}{9}$ The points are (6,2,7) and $\left(-\frac{34}{9}, -\frac{26}{9}, -\frac{25}{9}\right)$

28. (a) For the foot P, $DP \perp line \Rightarrow \begin{pmatrix} 2+2\lambda \\ \lambda-1 \\ 3+2\lambda \end{pmatrix} \cdot \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix} = 0 \Rightarrow \lambda = -1$. The point is (0, –1,1)

(b) For P(0, –1,1) (from (e), $d(D,P) = \sqrt{5}$

(c) For D(0,1,0), P(0, –1,1) it is D′(0, –3,2) (since P is the midpoint of DD′)

29. **EITHER**

The position vector for the point nearest to the origin is perpendicular to the direction of the line. At that point:

$$\begin{pmatrix} 1-\lambda \\ 2-3\lambda \\ 2 \end{pmatrix} \bullet \begin{pmatrix} -1 \\ -3 \\ 0 \end{pmatrix} = 0 \Rightarrow 10\lambda - 7 = 0 \Rightarrow \lambda = \frac{7}{10}$$

OR

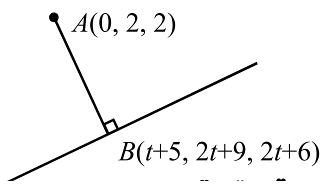
Let s be the distance from the origin to a point on the line, then

$s^2 = (1-\lambda)^2 + (2-3\lambda)^2 + 4 = 10\lambda^2 - 14\lambda + 9$

$$\frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 20\lambda - 14 \quad \text{For minimum } \frac{\mathrm{d}(s^2)}{\mathrm{d}\lambda} = 0, \Rightarrow \lambda = \frac{7}{10}$$

THEN $x = \frac{3}{10}, y = -\frac{1}{10}$ The point is $\left(\frac{3}{10}, \frac{-1}{10}, 2\right)$.

30. (a)



$$\overrightarrow{AB} \cdot \vec{v} = 0 \Rightarrow \begin{pmatrix} t+5 \\ 2t+7 \\ 2t+4 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 2 \\ 2 \end{pmatrix} = 0 \Rightarrow t + 5 + 4t + 14 + 4t + 8 = 0 \Rightarrow t = -3$$

We obtain $B(2,3,0)$

(b) Then $\overrightarrow{AB} = \begin{pmatrix} 2 \\ 1 \\ -2 \end{pmatrix}$, and the required distance is $|AB| = 3$.

(c) B (2,3,0) is the midpoint of AA'. Since we know $A(0,2,2)$, we obtain $A'(4,4,-2)$

31. (a) Using direction vectors $\boldsymbol{u} = \begin{pmatrix} -2 \\ 6 \\ 10 \end{pmatrix}$ and $\boldsymbol{v} = \begin{pmatrix} -6 \\ 10 \\ -2 \end{pmatrix}$

$$\cos\theta = \frac{52}{\sqrt{140}\sqrt{140}} = \frac{52}{140}$$

(b) (i) For substituting $s = 1$, position vector of P is $\begin{pmatrix} 7 \\ 10 \\ 4 \end{pmatrix}$

(ii) $\begin{pmatrix} 7 \\ 10 \\ 4 \end{pmatrix} = \begin{pmatrix} 1 \\ 20 \\ 2 \end{pmatrix} + t\begin{pmatrix} -6 \\ 10 \\ -2 \end{pmatrix}$

$7 = 1 - 6t \Rightarrow t = -1$

verify for 2nd coordinate, $10 = 20 + (-1)(10)$ and 3rd coordinate, $4 = 2 + (-1)(-2)$

Thus, P is also on L_2.

(c) $k\begin{pmatrix} -2 \\ 6 \\ 10 \end{pmatrix} = \begin{pmatrix} 6 \\ x \\ -30 \end{pmatrix}$

$-2k = 6 \Rightarrow k = -3$

$x = -3 \times 6 = -18$

B. Paper 2 questions (*LONG*)

32. (a) (i) $\vec{AB} = \vec{OB} - \vec{OA} = \begin{pmatrix}17\\2\\5\end{pmatrix} - \begin{pmatrix}7\\-3\\-5\end{pmatrix} = \begin{pmatrix}10\\5\\10\end{pmatrix}$

(ii) $\vec{AB} = \sqrt{10^2 + 5^2 + 10^2} = 15$

(b) (i) $\vec{AB}\cdot\vec{AE} = 0 \ \ ((-6)(-2) + 6(-4) + 3(4))$

$\vec{AB}\cdot\vec{AD} = 0 \ \ ((10)(-6) + 5(6) + 10(3))$

$\vec{AB}\cdot\vec{AE} = 0 \ \ ((10)(-2) + 5(-4) + 10(4))$

(ii) $90° \left(\text{or } \frac{\pi}{2}\right)$

(c) Volume $= |\vec{AB}| \times |\vec{AD}| \times |\vec{AE}| = 15 \times 9 \times 6 = 810$ (cubic units)

(d) Setting up a valid equation involving H. There are many possibilities.

eg $\begin{pmatrix}x-9\\y-4\\z-12\end{pmatrix} = \begin{pmatrix}-10\\-5\\-10\end{pmatrix}$

coordinates of H are (– 1, –1, 2)

(e) $\vec{HB} = \begin{pmatrix}18\\3\\3\end{pmatrix}$

$\cos\hat{P} = \frac{\vec{AG}\cdot\vec{HB}}{|\vec{AG}||\vec{HB}|} = \frac{2\times18+7\times3+17\times3}{\sqrt{2^2+7^2+17^2}\sqrt{18^2+3^2+3^2}} \left(= \frac{108}{\sqrt{342}\sqrt{342}}\right) = 0.31578...$

$\hat{P} = 71.6°$ (= 1.25 radians)

33. (a) (i) (JQ) $\boldsymbol{r} = \begin{pmatrix}6\\0\\0\end{pmatrix} + t\begin{pmatrix}-6\\7\\10\end{pmatrix}$ $\left(\text{or } \boldsymbol{r} = \begin{pmatrix}0\\7\\10\end{pmatrix} + t\begin{pmatrix}-6\\7\\10\end{pmatrix}\right)$

(ii) (MK) $\boldsymbol{r} = \begin{pmatrix}0\\7\\0\end{pmatrix} + s\begin{pmatrix}6\\-7\\10\end{pmatrix}$.

(b) $\cos\theta = \frac{15}{\sqrt{185}\sqrt{185}} \left(= \frac{15}{185} = 0.0811\right) \Rightarrow \theta = 1.49$ (radians), 85.3°

(c) **METHOD 1** Geometric approach

Valid reasoning *eg* diagonals bisect each other, $\vec{OD} = \vec{OM} + \frac{1}{2}\vec{MK}$

Calculation of mid point *eg* $\left(\frac{6+0}{2}, \frac{0+7}{2}, \frac{0+10}{2}\right)$

$\vec{OD} = \begin{pmatrix}3\\3.5\\5\end{pmatrix}$ (accept (3,3.5,5))

METHOD 2

$$\begin{pmatrix}6\\0\\0\end{pmatrix}+\boldsymbol{t}\begin{pmatrix}-6\\7\\10\end{pmatrix}=\begin{pmatrix}0\\7\\0\end{pmatrix}+\boldsymbol{s}\begin{pmatrix}6\\-7\\10\end{pmatrix}$$

$6 - 6t = 6s$,
$7t = 7 - 7s$,
$10t = 10s$
$\Rightarrow s = 0.5 \;\; t = 0.5$

$$\overrightarrow{OD}=\begin{pmatrix}3\\3.5\\5\end{pmatrix}$$ (accept (3, 3.5, 5))

METHOD 3

$$\begin{pmatrix}0\\7\\10\end{pmatrix}+t\begin{pmatrix}-6\\7\\10\end{pmatrix}=\begin{pmatrix}0\\7\\0\end{pmatrix}+s\begin{pmatrix}6\\-7\\10\end{pmatrix}$$

$-6t = 6s$,
$7 + 7t = 7 - 7s$,
$10 + 10t = 10s$
$\Rightarrow s = 0.5 \;\; t = -0.5$

$$\overrightarrow{OD}=\begin{pmatrix}3\\3.5\\5\end{pmatrix}$$ (accept (3, 3.5, 5))

34. (a) (i) $\overrightarrow{BC}=\overrightarrow{OC}-\overrightarrow{OB}=-6\boldsymbol{i}-2\boldsymbol{j}=\begin{pmatrix}-6\\-2\end{pmatrix}$ (ii) $\overrightarrow{OD}=\overrightarrow{OA}+\overrightarrow{BC}=-2\boldsymbol{i}+0\boldsymbol{j}\ (=-2\boldsymbol{i})=\begin{pmatrix}-2\\0\end{pmatrix}$

(b) $\overrightarrow{BD}=\overrightarrow{OD}-\overrightarrow{OB}=-3\boldsymbol{i}+3\boldsymbol{j}=\begin{pmatrix}-3\\3\end{pmatrix}$ and $\overrightarrow{AC}=\overrightarrow{OC}-\overrightarrow{OA}=-9\boldsymbol{i}-7\boldsymbol{j}=\begin{pmatrix}-9\\-7\end{pmatrix}$

Let θ be the angle between $\overrightarrow{BD}$ and $\overrightarrow{AC}$

therefore, $\cos\theta=\dfrac{6}{\sqrt{2340}} \Rightarrow \theta=82.9^\circ$ (1.45 rad)

(c) $\boldsymbol{r}=\boldsymbol{i}-3\boldsymbol{j}+\boldsymbol{t}(2\boldsymbol{i}+7\boldsymbol{j})$ **OR** $\boldsymbol{r}=\begin{pmatrix}1\\-3\end{pmatrix}+s\begin{pmatrix}2\\7\end{pmatrix}$

(d) **EITHER**

$4\boldsymbol{i}+2\boldsymbol{j}+s(\boldsymbol{i}+4\boldsymbol{j})=\boldsymbol{i}-3\boldsymbol{j}+t(2\boldsymbol{i}+7\boldsymbol{j})$

$$\left.\begin{aligned}4+s&=1+2t\\2+4s&=-3+7t\end{aligned}\right\}\Rightarrow t=7 \text{ and/or } s=11$$

Position vector of P is $15\boldsymbol{i}+46\boldsymbol{j}=\begin{pmatrix}15\\46\end{pmatrix}$

OR

$7x-2y=13$ or equivalent
$4x-y=14$ or equivalent
$x=15$, $y=46$

Position vector of P is $15\boldsymbol{i}+46\boldsymbol{j}=\begin{pmatrix}15\\46\end{pmatrix}$

35. (a) (i) $\vec{AB} = \begin{pmatrix} -4 \\ 6 \\ -1 \end{pmatrix}$

(ii) $\cos\theta = \dfrac{(-1)(-4)+(2)(6)+(-3)(-1)}{\sqrt{(-1)^2+2^2+(-3)^2}\sqrt{(-4)^2+6^2+(-1)^2}}, \dfrac{19}{\sqrt{14}\sqrt{53}}, 0.69751...$

$B\hat{A}O = 0.799$ radians (accept 45.8°)

(b) two correct answers: *e.g.* (1, – 2, 3), (–3, 4, 2), (–7, 10, 1), (–11, 16, 0)

(c) (i) $\boldsymbol{r} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

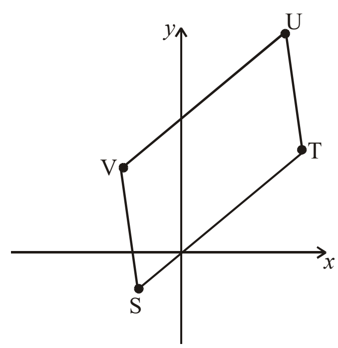
(ii) C on L_2, so $\begin{pmatrix} k \\ -k \\ 5 \end{pmatrix} = \begin{pmatrix} 1 \\ -2 \\ 3 \end{pmatrix} + t\begin{pmatrix} -3 \\ 4 \\ 2 \end{pmatrix}$

$1 - 3t = k$,
$-2 + 4t = -k$,
$5 = 3 + 2t$
$\Rightarrow t = 1, k = -2$
coordinates of C are (–2, 2, 5)

(d) $\begin{pmatrix} -2 \\ 2 \\ 5 \end{pmatrix} = \begin{pmatrix} 3 \\ -8 \\ 0 \end{pmatrix} + p\begin{pmatrix} 1 \\ -2 \\ -1 \end{pmatrix}$

$3 + p = -2$ **OR** $-8 - 2p = 2$ **OR** $-p = 5$
$p = -5$

36.



(a) $\overrightarrow{ST} = \boldsymbol{t} - \boldsymbol{s} = \begin{pmatrix} 7 \\ 7 \end{pmatrix} - \begin{pmatrix} -2 \\ -2 \end{pmatrix} = \begin{pmatrix} 9 \\ 9 \end{pmatrix}$

$\overrightarrow{VU} = \overrightarrow{ST}$

$\boldsymbol{u} - \boldsymbol{v} = \begin{pmatrix} 9 \\ 9 \end{pmatrix} \Rightarrow \boldsymbol{v} = \boldsymbol{u} - \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} - \begin{pmatrix} 9 \\ 9 \end{pmatrix} = \begin{pmatrix} -4 \\ 6 \end{pmatrix}$

V(–4, 6)

(b) Equation of (UV): direction is $= \begin{pmatrix} 9 \\ 9 \end{pmatrix}\left(\text{or } k\begin{pmatrix} 1 \\ 1 \end{pmatrix}\right)$

$\boldsymbol{r} = \begin{pmatrix} 5 \\ 15 \end{pmatrix} + \lambda\begin{pmatrix} 9 \\ 9 \end{pmatrix}$ or $\begin{pmatrix} 5 \\ 15 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$ or $\boldsymbol{r} = \begin{pmatrix} -4 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 9 \\ 9 \end{pmatrix}$ or $\begin{pmatrix} -4 \\ 6 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ 1 \end{pmatrix}$

(c) $\begin{pmatrix}1\\11\end{pmatrix}$ is on the line because it gives the same value of λ, for both the x and y coordinates.

For example, $1 = 5 + 9\lambda \quad \lambda = -\frac{4}{9}$

$11 = 15 + 9\lambda \quad \lambda = -\frac{4}{9}$

(d) (i) $\overrightarrow{EW} = \begin{pmatrix}a\\17\end{pmatrix} - \begin{pmatrix}1\\11\end{pmatrix} = \begin{pmatrix}a-1\\6\end{pmatrix}$

$|\overrightarrow{EW}| = 2\sqrt{13} \Rightarrow \sqrt{(a-1)^2 + 36} = 2\sqrt{13}$ (or $(a-1)^2 + 36 = 52$)

$(a-1)^2 = 16$

$a - 1 = 4$ or $a - 1 = -4$

$a = 5$ or $a = -3$

(ii) For $a = -3$

$\overrightarrow{EW} = \begin{pmatrix}-4\\6\end{pmatrix} \quad \overrightarrow{ET} = \boldsymbol{t} - \boldsymbol{e} = \begin{pmatrix}6\\-4\end{pmatrix}$

$\cos \hat{WET} = \dfrac{\overrightarrow{EW}\cdot\overrightarrow{ET}}{|\overrightarrow{EW}||\overrightarrow{ET}|} = \dfrac{-24-24}{\sqrt{52}\sqrt{52}} = -\dfrac{12}{13}$

Therefore, $\hat{WET} = 157°$ (3 sf)

37. (a) (i) $\overrightarrow{PQ} = \overrightarrow{OQ} - \overrightarrow{OP} = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k} = \begin{pmatrix}1\\-2\\3\end{pmatrix}$

(ii) $\boldsymbol{r} = \overrightarrow{OP} + s\overrightarrow{PQ} = -5\boldsymbol{i} + 11\boldsymbol{j} - 8\boldsymbol{k} + s(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k})$

$= (-5 + s)\,\boldsymbol{i} + (11 - 2s)\boldsymbol{j} + (-8 + 3s)\,\boldsymbol{k}$

(b) If $(2, y_1, z_1)$ lies on L_1 then $-5 + s = 2 \Rightarrow s = 7$

$y_1 = -3,\ z_1 = 13$

(c) $-5\boldsymbol{i} + 11\boldsymbol{j} - 8\boldsymbol{k} + s(\boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}) = 2\boldsymbol{i} + 9\boldsymbol{j} + 13\boldsymbol{k} + t(\boldsymbol{i} + 2\boldsymbol{j} + 3\boldsymbol{k})$

$-5 + s = 2 + t,$

$11 - 2s = 9 + 2t,$

$-8 + 3s = 13 + 3t$

$(s = 4, t = -3)$

$\overrightarrow{OT} = -\boldsymbol{i} + 3\boldsymbol{j} + 4\boldsymbol{k}$

(d) Direction vector for L_1 is $\boldsymbol{d}_1 = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

Direction vector for L_2 is $\boldsymbol{d}_2 = \boldsymbol{i} - 2\boldsymbol{j} + 3\boldsymbol{k}$

$\boldsymbol{d}_1 \bullet \boldsymbol{d}_2 = 6,\ |\mathrm{d}_1| = \sqrt{14},\ |\mathrm{d}_2| = \sqrt{14},$

$\cos\theta = \dfrac{6}{\sqrt{14}\sqrt{14}} \left(= \dfrac{6}{14} = \dfrac{3}{7}\right) \Rightarrow \theta = 64.6°(= 1.13\text{radians})$

38. (a) (i) $\vec{AB} = \vec{OB} - \vec{OA} = (4\boldsymbol{i} - 5\boldsymbol{j} + 21\boldsymbol{k}) - (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) = 2\boldsymbol{i} - 8\boldsymbol{j} + 20\boldsymbol{k}$

(ii) $|\vec{AB}| = \sqrt{2^2 + (-8)^2 + 20^2}\ \left(= \sqrt{468} = 6\sqrt{13} = 2\sqrt{117} = 21.6\right)$

$\boldsymbol{u} = \frac{1}{\sqrt{468}}(2\boldsymbol{i} - 8\boldsymbol{j} + 20\boldsymbol{k})\ \ \left(= \frac{2}{\sqrt{468}}\boldsymbol{i} - \frac{8}{\sqrt{468}}\boldsymbol{j} + \frac{20}{\sqrt{468}}\boldsymbol{k}\right)$

(iii) If the scalar product is zero, the vectors are perpendicular.

Finding an appropriate scalar product $\left(\boldsymbol{u} \bullet \vec{OA} \text{ or } \vec{AB} \bullet \vec{OA}\right)$

eg $\boldsymbol{u} \bullet \vec{OA} = \left(\frac{2}{\sqrt{468}}\right) \times 2 + \left(\frac{-8}{\sqrt{468}}\right) \times 3 + \left(\frac{20}{\sqrt{468}}\right) \times 1\ \left(= \frac{4 - 24 + 20}{\sqrt{468}}\right) = 0$

or $\vec{AB} \bullet \vec{OA} = 2 \times 2 + (-8) \times 3 + 20 \times 1 = 0$

(b) (i) **EITHER**

$S\left(\frac{2+4}{2}, \frac{3-5}{2}, \frac{1+21}{2}\right)$ Therefore, $\vec{OS} = 3\boldsymbol{i} - \boldsymbol{j} + 11\boldsymbol{k}$ (accept (3, −1, 11))

OR

$\vec{OS} = \vec{OA} + \frac{1}{2}\vec{AB} = (2\boldsymbol{i} + 3\boldsymbol{j} + \boldsymbol{k}) + \frac{1}{2}(2\boldsymbol{i} + 8\boldsymbol{j} + 20\boldsymbol{k}) = 3\boldsymbol{i} - \boldsymbol{j} + 11\boldsymbol{k}$

(ii) $L_1 : r = (3\boldsymbol{i} - \boldsymbol{j} + 11\boldsymbol{k}) + t\,(2\boldsymbol{i} + 3\boldsymbol{j} + 1\boldsymbol{k})$ **OR** $\boldsymbol{r} = \begin{pmatrix} 3 \\ -1 \\ 11 \end{pmatrix} + s\begin{pmatrix} 2 \\ 3 \\ 1 \end{pmatrix}$.

(c) Using direction vectors (*eg* $2\boldsymbol{i} + 3\boldsymbol{j} + 1\boldsymbol{k}$ and $-2\boldsymbol{i} + 5\boldsymbol{j} - 3\boldsymbol{k}$)
Direction vectors are not scalar multiples of each other.

(d) $3 + 2t = 5 - 2s$,
$-1 + 3t = 10 + 5s \quad \Rightarrow (s = -1, t = 2)$
$11 + t = 10 - 3s$
P has position vector $7\boldsymbol{i} + 5\boldsymbol{j} + 13\boldsymbol{k}$

39. (a) 27.3
(b) 48.2
(c) P(5,6,7)
(d) The foot is P(29/9, 22/9, 31/9) so the reflection of A is A′(49/9,8/9,35/9)
(e) it is the line passing through P(5,6,7) and A′(49/9,8/9,35/9)

$$\boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 4/9 \\ -46/9 \\ -28/9 \end{pmatrix} \text{ which can be simplified to } \boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 2 \\ -23 \\ -14 \end{pmatrix}$$

(f) (i) $|\boldsymbol{d_1}| = 3,\ |\boldsymbol{d_2}| = 3$, (ii) $\boldsymbol{d_1} + \boldsymbol{d_2} = \begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix},\ \boldsymbol{d_1} - \boldsymbol{d_2} = \begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$

(iii) They pass through P(5,6,7) . Since the direction vectors have the same magnitude, for the bisector lines we can consider the direction vectors $\boldsymbol{d_1} \pm \boldsymbol{d_2}$

$$\boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 3 \\ 3 \\ 4 \end{pmatrix} \text{ and } \boldsymbol{r} = \begin{pmatrix} 5 \\ 6 \\ 7 \end{pmatrix} + \lambda\begin{pmatrix} 1 \\ -1 \\ 0 \end{pmatrix}$$

(otherwise we would add and subtract the unit vectors $\widehat{\boldsymbol{d_1}}$ and $\widehat{\boldsymbol{d_2}}$)

40. (a) Given the points A(–1, 2, 3), B(–1, 3, 5) and C(0, –1, 1),

then $\overrightarrow{AB} = \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix}, \overrightarrow{AC} = \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix}$ and $\left|\overrightarrow{AB}\right| = \sqrt{5}, \left|\overrightarrow{AC}\right| = \sqrt{14}$

$$\cos\theta = \left(\frac{\overrightarrow{AB}\cdot\overrightarrow{AC}}{\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right|}\right) = \frac{-7}{\sqrt{5}\sqrt{14}} \Rightarrow \theta = 147° \text{ (3 s.f.) or } 2.56 \text{ rad}$$

(b) Area $= \frac{1}{2}\left|\overrightarrow{AB}\right|\left|\overrightarrow{AC}\right| \sin\theta$ or $\frac{1}{2}\left|\overrightarrow{AB}\times\overrightarrow{AC}\right| = 2.29 \text{ units}^2 \quad (= \frac{\sqrt{21}}{2})$

(c) (i) The parametric equations of l_1 and l_2 are

l_1: $x = 2$, $y = -1 + \lambda$, $z = 2\lambda$

l_2: $x = -1 + \mu$, $y = 1 - 3\mu$, $z = 1 - 2\mu$

(ii) To test for a point of intersection we use the system of equations:

$2 = -1 + \mu$ (1)

$-1 + \lambda = 1 - 3\mu$ (2)

$2\lambda = 1 - 2\mu$ (3)

Then $\mu = 3$, $\lambda = -7$ from (1) and (2)

Substituting into (3) gives RHS = –14, LHS = –5

Therefore the system of equations has no solution and the lines do not intersect.

(d) Consider random points $P(2,-1+\lambda,2\lambda)$ on l_1 and $Q(-1+\mu,1-3\mu, 1-2\mu)$ on l_2.

Then $PQ = \begin{pmatrix} -3+\mu \\ 2-3\mu-\lambda \\ 1-2\mu-2\lambda \end{pmatrix}$.

PQ is perpendicular to both lines: so $PQ \cdot \begin{pmatrix} 0 \\ 1 \\ 2 \end{pmatrix} = 0$ and $PQ \cdot \begin{pmatrix} 1 \\ -3 \\ -2 \end{pmatrix} = 0$

That is

$$(2-3\mu-\lambda) + 2(1-2\mu-2\lambda) = 0 \Rightarrow 5\lambda + 7\mu = 4$$

$$(-3+\mu) - 3(2-3\mu-\lambda) - 2(1-2\mu-2\lambda) = 0 \Rightarrow 7\lambda + 14\mu = 11$$

This gives $\lambda = -1$, $\mu = 9/7$. $PQ = \begin{pmatrix} -12/7 \\ -6/7 \\ 3/7 \end{pmatrix}$. $|PQ| = \frac{\sqrt{189}}{7} = 1.96$

41. (a) $\overrightarrow{AB} = \boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k}$ $\quad \left|\overrightarrow{AB}\right| = \sqrt{3}$ **and** $\left|\overrightarrow{OA}\right| = 3\sqrt{2}$, $\quad \overrightarrow{OA} \bullet \overrightarrow{AB} = 6$

$$\cos\theta = \frac{\overrightarrow{OA} \bullet \overrightarrow{AB}}{\left|\overrightarrow{OA}\right|\left|\overrightarrow{AB}\right|} = \frac{2}{\sqrt{6}} \left(= \frac{\sqrt{6}}{3}\right)$$

(b) L_1: $\boldsymbol{r} = \overrightarrow{OA} + s\overrightarrow{AB} = \boldsymbol{i} - \boldsymbol{j} + 4\boldsymbol{k} + s(\boldsymbol{i} - \boldsymbol{j} + \boldsymbol{k})$ or equivalent

(c) $1 + s = 2 + 2t$,

$-1 - s = 4 + t$,

$4 + s = 7 + 3t$

Finding either $s = -3$ or $t = -2$

Explicitly showing that these values satisfy the third equation

Point of intersection is (–2, 2, 1)