EXERCISES [MAI 1.9-1.11]

COMPLEX NUMBERS (POLAR FORM)

SOLUTIONS

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A. Paper 1 questions (SHORT)

1.

Cartesian form $x+y\mathrm{i}$	Polar form		
	$r\mathrm{cis}\theta$	$r(\cos\theta+\mathrm{i}\sin\theta)$	$r\mathrm{e}^{\mathrm{i}\theta}$
1	$\mathrm{cis}0$	$\cos0+\mathrm{i}\sin0$	$\mathrm{e}^{0\mathrm{i}}$
-1	$\mathrm{cis}\pi$	$\cos\pi+\mathrm{i}\sin\pi$	$\mathrm{e}^{\pi\mathrm{i}}$
i	$\mathrm{cis}\frac{\pi}{2}$	$\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}$	$\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$
$-\mathrm{i}$	$\mathrm{cis}\frac{3\pi}{2}$ or $\mathrm{cis}(-\frac{\pi}{2})$	$\cos(-\frac{\pi}{2})+\mathrm{i}\sin(-\frac{\pi}{2})$	$\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}$
8	$8\mathrm{cis}0$	$8(\cos0+\mathrm{i}\sin0)$	$\mathrm{e}^{0\mathrm{i}}$
-8	$8\mathrm{cis}\pi$	$8(\cos\pi+\mathrm{i}\sin\pi)$	$\mathrm{e}^{\pi\mathrm{i}}$
$8\mathrm{i}$	$8\mathrm{cis}\frac{\pi}{2}$	$8\left(\cos\frac{\pi}{2}+\mathrm{i}\sin\frac{\pi}{2}\right)$	$\mathrm{e}^{\frac{\pi}{2}\mathrm{i}}$
$-8\mathrm{i}$	$8\mathrm{cis}\frac{3\pi}{2}$ or $\mathrm{cis}(-\frac{\pi}{2})$	$8\left(\cos(-\frac{\pi}{2})+\mathrm{i}\sin(-\frac{\pi}{2})\right)$	$\mathrm{e}^{-\frac{\pi}{2}\mathrm{i}}$

2.

Polar form	Cartesian form
$4\mathrm{cis}0$	4
$5\mathrm{cis}\pi$	-5

Polar form	Cartesian form
$6\mathrm{cis}\frac{\pi}{2}$	6i
$7\mathrm{cis}\left(-\frac{\pi}{2}\right)$	-7i

3. $z_1=1+\mathrm{i}=\sqrt{2}\mathrm{cis}\frac{\pi}{4}$, $\quad z_2=-1+\mathrm{i}=\sqrt{2}\mathrm{cis}\frac{3\pi}{4}$,

$z_3=-1-\mathrm{i}=\sqrt{2}\mathrm{cis}(-\frac{3\pi}{4})$, $\quad z_4=1-\mathrm{i}=\sqrt{2}\mathrm{cis}(-\frac{\pi}{4})$

They lie on a circle of radius $\sqrt{2}$, with arguments $\frac{\pi}{4}$, $\frac{3\pi}{4}$, $-\frac{3\pi}{4}$, $-\frac{\pi}{4}$.

OR on the Cartesian plane: dots at points (1,1), (–1,1), (–1, –1), (1, –1) respectively.

4. $z = \sqrt{3} + \mathrm{i} = 2\mathrm{cis}\frac{\pi}{6}$, $\quad -z = -\sqrt{3} - \mathrm{i} = 2\mathrm{cis}(\frac{-5\pi}{6})$,

$\bar{z} = \sqrt{3} - \mathrm{i} = 2\mathrm{cis}(\frac{-\pi}{6})$, $\quad -\bar{z} = -\sqrt{3} + \mathrm{i} = 2\mathrm{cis}(\frac{5\pi}{6})$

They lie on a circle of radius 2, with arguments $\frac{\pi}{6}, \frac{-5\pi}{6}, -\frac{\pi}{6}, \frac{5\pi}{6}$

OR on the Cartesian plane: dots at points ($\sqrt{3}$,1), ($-\sqrt{3}$,–1), ($\sqrt{3}$,–1), ($-\sqrt{3}$,1) respectively.

5. (a) For $z = 2\mathrm{cis}20°$ and $w = 6\mathrm{cis}40°$

zw	$12\mathrm{cis}60°$
$\frac{w}{z}$	$3\mathrm{cis}20°$
$\frac{z}{w}$	$\frac{1}{3}\mathrm{cis}(-20°)$
z^2	$4\mathrm{cis}40°$
z^3	$8\mathrm{cis}60°$

(b) $zw + z^3 = 20\mathrm{cis}60° = 20(\cos 60° + \mathrm{i}\sin 60°) = 10 + 10\sqrt{3}\mathrm{i}$

6. **EITHER directly by GDC:** $z = 8\mathrm{cis}60°$ (or $z = 8\mathrm{cis}\frac{\pi}{3}$)

OR analytically

$$z = 4\left(-\frac{1}{2} + \mathrm{i}\frac{\sqrt{3}}{2}\right) + 4\sqrt{3}\left(\frac{\sqrt{3}}{2} + \mathrm{i}\frac{1}{2}\right) = -2 + 2\mathrm{i}\sqrt{3} + 6 + 2\mathrm{i}\sqrt{3} = 4 + 4\mathrm{i}\sqrt{3}$$

$r = 8 \quad \theta = 60°$, $\quad z = 8\mathrm{cis}60°$ (or $z = 8\mathrm{cis}\frac{\pi}{3}$ or $z = 8\mathrm{e}^{\mathrm{i}\frac{\pi}{3}}$)

7. (a) $z = 2\mathrm{cis}60°$ $\quad w = \mathrm{cis}(-45°)$ $\quad u = 4\mathrm{cis}150°$

(b) (i) $zw = 2\mathrm{cis}15°$ (ii) $\frac{z}{w} = 2\mathrm{cis}105°$ (iii) $u^2 = 16\mathrm{cis}300°$ (iv) $zwu = 8\mathrm{cis}165°$

8. (a) $\frac{z_1}{z_2} = \left(\frac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2}\right)\left(\frac{1}{1-\mathrm{i}}\right)\left(\frac{1+\mathrm{i}}{1+\mathrm{i}}\right) = \frac{\sqrt{6} + \sqrt{2} + \mathrm{i}(\sqrt{6} - \sqrt{2})}{4}$ (or directly by GDC)

(b) $z_1 = \frac{\sqrt{6} - \mathrm{i}\sqrt{2}}{2} = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{12}\right)$ $\quad z_2 = 1 - \mathrm{i} = \sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)$

(c) $\frac{z_1}{z_2} = \frac{\sqrt{2}\mathrm{cis}\left(-\frac{\pi}{6}\right)}{\sqrt{2}\mathrm{cis}\left(-\frac{\pi}{4}\right)} = 1\mathrm{cis}\left(-\frac{\pi}{6} + \frac{\pi}{4}\right) = \mathrm{cis}\frac{\pi}{12} = \cos\frac{\pi}{12} + \mathrm{i}\sin\frac{\pi}{12}$

(d) From (a) and (c), Real parts: $\cos\frac{\pi}{12} = \frac{\sqrt{6} + \sqrt{2}}{4}$, Imaginary parts: $\sin\frac{\pi}{12} = \frac{\sqrt{6} - \sqrt{2}}{4}$

9. The modulus is $\dfrac{a^2}{b^2}$, the argument is $2\times\dfrac{3\pi}{4}-2\times\pi=\dfrac{3\pi}{2}-2\pi=-\dfrac{\pi}{2}$

Hence $\left(\dfrac{z_2}{z_1}\right)^2=\dfrac{a^2}{b^2}\text{cis}\left(-\dfrac{\pi}{2}\right)=-\dfrac{a^2}{b^2}\text{i}$

10. $z=r\text{cis}\theta$, $w=3\text{cis}\dfrac{\pi}{5}$

(a) (i) $zw=3r\text{cis}(\theta+\dfrac{\pi}{5})$ (ii) $\dfrac{z}{w}=\dfrac{r}{3}\text{cis}(\theta-\dfrac{\pi}{5})$ (ii) $zw^2=9r^2\text{cis}(\theta+\dfrac{2\pi}{5})$

(b) (i) enlargement by scale factor 3, anticlockwise rotation by $\dfrac{\pi}{5}$

(ii) enlargement by scale factor $\dfrac{r}{3}$, clockwise rotation by $\dfrac{\pi}{5}$

(iii) enlargement by scale factor 9, anticlockwise rotation by $\dfrac{2\pi}{5}$

11. **METHOD A**

$(b+\text{i})^2=b^2-1+2b\text{i}$

$\tan 60°=\dfrac{2b}{b^2-1}\Leftrightarrow\dfrac{2b}{b^2-1}=\sqrt{3}\Leftrightarrow\sqrt{3}b^2-\sqrt{3}=2b\Leftrightarrow\sqrt{3}b^2-2b-\sqrt{3}=0$

Roots: $\sqrt{3}$ and $-\dfrac{1}{\sqrt{3}}$ (rejected)

METHOD B

$\arg(b+\text{i})^2=60°\Rightarrow\arg(b+\text{i})=30°$

$\tan 30°=\dfrac{1}{b}\Rightarrow\dfrac{1}{b}=\dfrac{1}{\sqrt{3}}\Rightarrow b=\sqrt{3}$

12. $zw=6\text{cis}(2x+3)=6\text{e}^{(2x+3)\text{i}}=6[\cos(2x+3)+\text{i}\sin(2x+3)]$

13. (a) $zw=16\text{cis}x=16\text{e}^{x\text{i}}=16(\cos x+\text{i}\sin x)$

(b) $\dfrac{w}{z}=4\text{cis}0.6x=4\text{e}^{0.6x\text{i}}=4(\cos 0.6x+\text{i}\sin 0.6x)$

14. (a) $z+w=2\text{e}^{(2x+1)\text{i}}+3\text{e}^{(2x+3)\text{i}}=2\text{e}^{2x\text{i}}\text{e}^{\text{i}}+3\text{e}^{2x\text{i}}\text{e}^{3\text{i}}=(2\text{e}^{\text{i}}+3\text{e}^{3\text{i}})\text{e}^{2x\text{i}}$

$=2.83\text{e}^{2.30\text{i}}\text{e}^{2x\text{i}}=2.83\text{e}^{(2x+2.30)\text{i}}$

(b) (i) Real part: $2\cos(2x+1)+3\cos(2x+3)=2.83\cos(2x+2.30)$

(ii) Imaginary part: $2\sin(2x+1)+3\sin(2x+3)=2.83\sin(2x+2.30)$

15. (a) $z+w=3\text{e}^{(0.2x+1)\text{i}}+5\text{e}^{0.2x\text{i}}=3\text{e}^{0.2x\text{i}}\text{e}^{\text{i}}+5\text{e}^{0.2x\text{i}}=(3\text{e}^{\text{i}}+5)\text{e}^{0.2x\text{i}}$

$=7.09\text{e}^{0.364\text{i}}\text{e}^{2x\text{i}}=7.09\text{e}^{(0.2x+0.364)\text{i}}$

(b) (i) Real part: $3\cos(0.2x+1)+5\cos(0.2x)=7.09\cos(0.2x+0.364)$

(ii) Imaginary part: $3\sin(0.2x+1)+5\sin(0.2x)=7.09\sin(0.2x+0.364)$

16. (a) $z_A = 10\text{cis}(3t+4) = 10\text{e}^{(3t+4)\text{i}}$ $\quad z_B = 20\text{cis}(3t+5) = 20\text{e}^{(3t+5)\text{i}}$

(b) $z_A + z_B = 10\text{e}^{(3t+4)\text{i}} + 20\text{e}^{(3t+5)\text{i}} = 10\text{e}^{3t\text{i}}\text{e}^{4\text{i}} + 20\text{e}^{3t\text{i}}\text{e}^{5\text{i}} = (10\text{e}^{4\text{i}} + 20\text{e}^{5\text{i}})\text{e}^{3t\text{i}}$

$= 26.8\text{e}^{-1.60\text{i}}\text{e}^{3t\text{i}} = 26.8\text{e}^{(3t-1.60)\text{i}}$

(c) The real part: $A(t) + B(t) = 26.8\cos(3t - 1.60)$

17. (a) Let $z_A = 7\text{cis}(0.7t) = 7\text{e}^{0.7t\text{i}}$ $\quad z_B = \text{cis}(0.7t+5) = \text{e}^{(0.7t+5)\text{i}}$

$z_A + z_B = 7\text{e}^{(0.7t)\text{i}} + \text{e}^{(0.7t+5)\text{i}} = 7\text{e}^{0.7t\text{i}} + \text{e}^{0.7t\text{i}}\text{e}^{5\text{i}} = (7 + \text{e}^{5\text{i}})\text{e}^{0.7t\text{i}}$

$= 7.35\text{e}^{-0,131\text{i}}\text{e}^{0.7t\text{i}} = 7.35\text{e}^{(0.7t-0.131)\text{i}}$

The imaginary part gives: $A(t) + B(t) = 7.35\sin(0.7t - 0.131)$

(b) We observe the graph of $A(t) + B(t) = 7\sin(0.7t) + \sin(0.7t+5)$

Amplitude: 7.35

Central value (increasing) at $t = 0.187$

Hence $A(t) + B(t) = 7.35\sin(0.7(t - 0.187)) = 7.35\sin(0.7t - 0.131)$

18. (a) Let $z_A = 5.1\text{cis}(\omega t) = 5.1\text{e}^{\omega t\text{i}}$ $\quad z_B = 3.2\text{cis}(\omega t+5) = \text{e}^{(\omega t+5)\text{i}}$

$z_A + z_B = 5.1\text{e}^{(\omega t)\text{i}} + 3.2\text{e}^{\omega t+5)\text{i}} = 5.1\text{e}^{\omega t\text{i}} + 3.2\text{e}^{\omega t\text{i}}\text{e}^{5\text{i}} = (5.1 + 3.2\text{e}^{5\text{i}})\text{e}^{\omega t\text{i}}$

$= 6.75\text{e}^{-0.472\text{i}}\text{e}^{\omega t\text{i}} = 6.75\text{e}^{(\omega t-0.472)\text{i}}$

The imaginary part gives: $A(t) + B(t) = 6.75\sin(\omega t - 0.472)$

(b) We observe the graph of $A(t) + B(t) = 5.1\sin x + 3.2\sin(x+5)$

Amplitude: 6.75

Central value (increasing) at $t = 0.472$

Hence $A(t) + B(t) = 6.75\sin(\omega t - 0.472)$

B. Paper 2 questions (LONG)

19. (a) $z = \sqrt{2}\text{cis}45^\circ$, $w = 2\text{cis(-}30^\circ)$

(b) $zw = (1+\text{i})\,(\sqrt{3} - \text{i}) = (\sqrt{3}+1) + (\sqrt{3}-1)\text{i}$

(c) $zw = 2\sqrt{2}\text{cis}15^\circ = 2\sqrt{2}\cos15^\circ + \text{i}2\sqrt{2}\sin15^\circ$

(d) Equal real parts: $\quad 2\sqrt{2}\cos15^\circ = \sqrt{3}+1 \Rightarrow \cos15^\circ = \dfrac{\sqrt{3}+1}{2\sqrt{2}} = \dfrac{\sqrt{6}+\sqrt{2}}{4}$

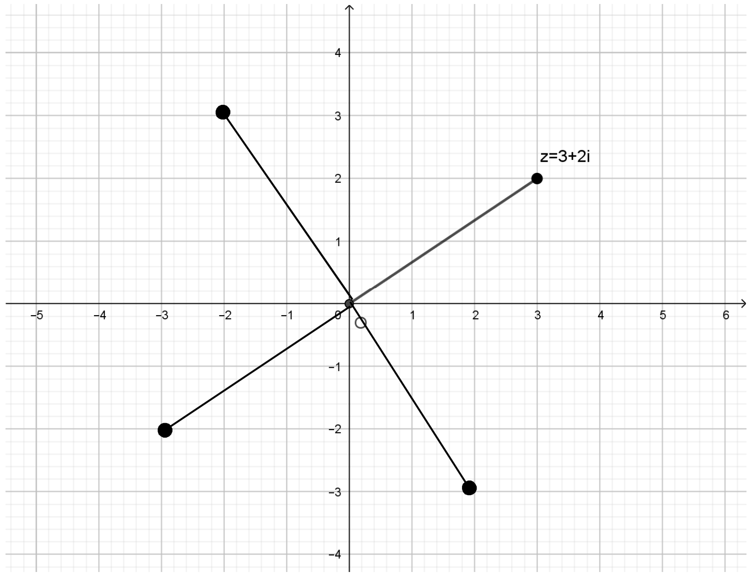
Equal imaginary parts: $\quad 2\sqrt{2}\sin15^\circ = \sqrt{3}-1 \Rightarrow \sin15^\circ = \dfrac{\sqrt{3}-1}{2\sqrt{2}} = \dfrac{\sqrt{6}-\sqrt{2}}{4}$

$$\tan15^\circ = \frac{\sin15^\circ}{\cos15^\circ} = \frac{\sqrt{3}-1}{\sqrt{3}+1} = 2 - \sqrt{3}$$

20. (a) $zi = -2 + 3i$,

$zi^2 = -3 - 2i$

$zi^3 = 2 - 3i$



(b) $r = \sqrt{13}$

(c) $zi = \sqrt{13}e^{i(\theta+\frac{\pi}{2})}$,

$zi^2 = \sqrt{13}e^{i(\theta+\pi)}$

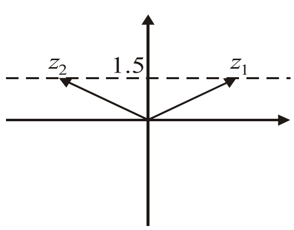
$zi^3 = \sqrt{13}e^{i(\theta+\frac{3\pi}{2})}$

z is rotated 3 times anticlockwise by $\frac{\pi}{2}$

21. (a) $z = x + yi \Rightarrow |x + yi| = |x + (y-3)i| \Rightarrow x^2 + y^2 = x^2 + (y-3)^2$

$\Rightarrow 6y - 9 = 0 \Rightarrow y = \dfrac{3}{2}$

(b) (i)



(ii) $\sin\theta_1 = \dfrac{1.5}{3} = \dfrac{1}{2} \Rightarrow \theta_1 = \dfrac{\pi}{6}$

(iii) $\theta_1 = \pi - \dfrac{\pi}{6} = \dfrac{5}{6}\pi$

(iv) $z_1 = 3\operatorname{cis}\dfrac{\pi}{6}$, $z_2 = 3\operatorname{cis}\dfrac{5\pi}{6}$

22. (a) $LHS = \frac{dy}{d\theta} = -\sin\theta + i\cos\theta$

$PHS = iy = i(\cos\theta + i\sin\theta) = i\cos\theta - \sin\theta$

Hence $LHS = RHS$

(b) When $\theta = 0$, $y = \cos 0 + i\sin 0 = 1$, $\frac{dy}{d\theta} = -\sin 0 + i\cos 0 = i$

(c)

$\int \frac{dy}{y} = i\int d\theta$

$\ln y = i\theta + c$

Substituting (0, 1) $0 = 0 + c \Rightarrow c = 0$

$\therefore \ln y = i\theta$

$y = e^{i\theta}$

(d) (i) $(\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi) = e^{i\theta}e^{i\varphi} = e^{i\theta + i\varphi} = e^{i(\theta+\varphi)} = \cos(\theta+\phi) + i\sin(\theta+\phi)$

(ii) $\frac{\cos\theta + i\sin\theta}{\cos\phi + i\sin\phi} = \frac{e^{i\theta}}{e^{i\varphi}} = e^{i\theta - i\varphi} = e^{i(\theta-\varphi)} = \cos(\theta-\phi) + i\sin(\theta-\phi)$

(iii) $(\cos\theta + i\sin\theta)^3 = (e^{i\theta})^3 = e^{i(3\theta)} = \cos 3\theta + i\sin 3\theta$