EXERCISES [MAI 1.7]

INFINITE SERIES

SOLUTIONS

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A. Paper 1 questions (SHORT)

1. (a) $10\times 0.5^{n-1}$ $(= 20\times 0.5^{n})$

(b) 0.000610

(c) 19.999981

(d) 20

2. (a) 5/3 (b) 5/7

3. (a) $\frac{4(4^6-1)}{4-1} = 5460$ (b) $\frac{0.25(1-0.25^6)}{1-0.25} =$ 0.333252 (c) $\frac{0.25}{1-0.25} = \frac{1}{3}$

4. (a) $u_{10} = 3(0.9)^9$

(b) $S = \frac{3}{1-0.9} = \frac{3}{0.1} = 30$

5. (a) $\frac{1}{5}$ $(= 0.2)$

(b) (i) $u_{10} = 25\left(\frac{1}{5}\right)^9 = 0.0000128$ (ii) $u_n = 25\left(\frac{1}{5}\right)^{n-1}$

(c) $S = \frac{125}{4} = 31.25$

6. (a) $r = \frac{2}{3}$

(b) $u_{15} = 1.39$

(c) $S = 1215$

7. r -$= -\frac{1800}{3000} = -0.6$ $S = \frac{3000}{1.6} = 1875$

8. (a) $r = \frac{16}{32}\left(= \frac{1}{2}\right)$

(b) $u_6 = 32\times\left(\frac{1}{2}\right)^{6-1} = 1$

(c) $S_\infty = \frac{32}{1-\frac{1}{2}} = 64$

9. $r = -\frac{1}{3}$ $S_\infty = \frac{27}{1+\frac{1}{3}} = \frac{81}{4}$ $(= 20.25)$

10. $S = \dfrac{u_1}{1-r} = \dfrac{\frac{2}{3}}{1-\left(-\frac{2}{3}\right)} = \dfrac{2}{3} \times \dfrac{3}{5} = \dfrac{2}{5}$

11. $u_3 = u_1 r^2 = 8 \Leftrightarrow 8 = 18r^2 \Leftrightarrow r^2 = \dfrac{8}{18}\left(=\dfrac{4}{9}\right) \Leftrightarrow r = \pm\dfrac{2}{3}$

$S_\infty = \dfrac{u_1}{1-r}$, $S_\infty = 54$ or $S_\infty = \dfrac{54}{5}$ (= 10.8)

12. $S_\infty = \dfrac{-12}{1-\left(-\frac{2}{3}\right)} = \dfrac{-36}{5}$ or – 7.2

13.

(a) $u_1 r^3 = -\dfrac{2}{3}$

$\Rightarrow 18r^3 = -\dfrac{2}{3}$

$\Rightarrow r^3 = -\dfrac{1}{27} \Rightarrow r = -\dfrac{1}{3}$

$S_n = \dfrac{18\left(1-\left(-\frac{1}{3}\right)^n\right)}{1-\left(-\frac{1}{3}\right)}$

$= \dfrac{27}{2}\left(1-\left(-\dfrac{1}{3}\right)^n\right)$

(b) $S_\infty = \dfrac{u_1}{1-r} = \dfrac{18}{1+\frac{1}{3}} = \dfrac{27}{2}$

14. $r = \dfrac{1}{2}$, $u_1 = 20$

15. $u_1 + u_1 r = 15 \Rightarrow u_1(1 + r) = 15$; $\dfrac{u_1}{1-r} = 27$

$1 - r^2 = \dfrac{15}{27}$

(i) $r = \dfrac{2}{3}$

(ii) $u_1 = 27 \times \dfrac{1}{3} = 9$

16. $S_\infty = \dfrac{u_1}{1-r} = 32$ and $S_4 = \dfrac{u_1(1-r^4)}{1-r} = 30$

Divide S_4 by S_∞ : $1 - r^4 = \dfrac{30}{32} = \dfrac{15}{16} \Leftrightarrow r^4 = \dfrac{1}{16} \Leftrightarrow r = \dfrac{1}{2}$ and so $u_1 = 16$

$S_\infty - S_8 = 32 - \dfrac{16(1-0.5^8)}{1-0.5} = 32 - 32(1-0.5^8) = 32 \times 0.5^8 = 0.0125$

17. $k+\frac{2}{3}k+\left(\frac{2}{3}\right)^2k+\left(\frac{2}{3}\right)^3k+\ldots\ldots\ldots=1$

$k\left(\dfrac{1}{1-\frac{2}{3}}\right)=1$ so $k=\frac{1}{3}$

18. (a) $\frac{x+1}{x-3}=\frac{2x+8}{x+1}\Leftrightarrow x^2+2x+1=2x^2+2x-24\Leftrightarrow x^2=25\Leftrightarrow x=5$ or $x=-5$

(b) For $x=-5$, $r=\frac{1}{2}$ so the series converges (while for $x=5$, $r=3$)

(c) $S=\dfrac{-8}{1-\frac{1}{2}}=-16$

19. (a) $-1<\frac{2x}{3}<1$. This gives $-1.5<x<1.5$ or $|x|<\frac{3}{2}$

(b) When $x=1.2$, the common ratio is $r=0.8$ and the sum is $\frac{1}{1-0.8}=5$

20. (a) $r=4-3x\Rightarrow -1<4-3x<1\Rightarrow 1<x<\frac{5}{3}$

(b) $x=1.2\Rightarrow u_1=0.8\ \ r=0.4$, $S_\infty=\frac{0.8}{0.6}=\frac{4}{3}$

B. Paper 2 questions (LONG)

21. (a) (i) Area B $=\frac{1}{16}$, area C = 64

(ii) $\dfrac{\frac{1}{16}}{\frac{1}{4}}=\frac{1}{4}$ $\dfrac{\frac{1}{64}}{\frac{1}{16}}=\frac{1}{4}$ (Ratio is the same.)

(iii) Common ratio $=\frac{1}{4}$

(b) (i) Total area (S_2) $=\frac{1}{4}+\frac{1}{16}=\frac{5}{16}$ = (= 0.3125) (0.313, 3 s.f.)

(ii) Required area $=S_8=\dfrac{\frac{1}{4}\left(1-\left(\frac{1}{4}\right)^8\right)}{1-\frac{1}{4}}=$ 0.333328 (6 s.f.)

(c) Sum to infinity $=\dfrac{\frac{1}{4}}{1-\frac{1}{4}}=\frac{1}{3}$

22. (a) (i) PQ = $\sqrt{AP^2 + AQ^2} = \sqrt{2^2 + 2^2} = \sqrt{4(2)} = 2\sqrt{2}$ cm

(ii) Area of PQRS = $(2\sqrt{2})(2\sqrt{2}) = 8$ cm^2

(b) (i) Side of third square = $\sqrt{\left(\sqrt{2}\right)^2 + \left(\sqrt{2}\right)^2} = \sqrt{4} = 2$ cm

Area of third square = 4 cm^2

(ii) $r = \frac{8}{16} = \frac{4}{8} = \frac{1}{2}$ $\Rightarrow$ geometric progression

(c) (i) $u_{11} = u_1 r^{10} = 16\left(\frac{1}{2}\right)^{10} = \frac{16}{1024} = \frac{1}{64}$ (= 0.015625 = 0.0156, 3 s.f.)

(ii) $S_\infty = \frac{u_1}{1-r} = \frac{16}{1-\frac{1}{2}} = 32$

23. (a) Let $p = 1.77$, $p = -0.5$, $p = -2.27$

(b) Geometric sequence: 4, $4p$, $4p^2$, $4p^3$

(c) Arithmetic sequence: 4, $4 + 5p$, $4 + 10p$, $4 + 15p$

(d) $4p^2 + 4p^3 = 4 + (4 + 15p) \Rightarrow 4p^3 + 4p^2 - 15p - 8 = 0$

So the values are as in (a): $p = 1.77$, $p = -0.5$, $p = -2.27$

(e) For $p = -0.5$, $S_\infty = \frac{4}{1-0.5} = 8$

24. (a) G.S. $u_1 = 1$, $r = e$ $S_{10} = \frac{e^{10}-1}{e-1}$

(b) G.S. $u_1 = 1$, $r = \frac{1}{e}$ $S_{10} = \frac{1-\frac{1}{e^{10}}}{1-\frac{1}{e}}$ $\left(= \frac{e^{10}-1}{e^{10}-e^9}\right)$

(c) $r = \frac{1}{e} < 1$ $S_\infty = \frac{1}{1-\frac{1}{e}} = \frac{e}{e-1}$

(d) $\frac{1}{1-e^x} = 2 \Leftrightarrow 1 - e^x = \frac{1}{2} \Leftrightarrow e^x = \frac{1}{2}$ (i) $x = \ln\frac{1}{2}$ ($= \ln 0.5$) (ii) $x = -0.693$

25. (a) The series in fact is $\log x + 2\log x + 3\log x + \cdots$

so there is a common difference $d = \log x$

(b) $u_1 = \log 3$, $d = \log 3$, $S_{10} = \frac{10}{2}\left(\log 3 + \log 3^{10}\right) = 5\left(11\log 3\right) = 55\log 3 = 26.2$

(c) There is a common ratio $r = \log x$

(d) $u_1 = \log 3$, $r = \log 3$, $S_{10} = \frac{\log 3(1-(\log 3)^{10})}{1-\log 3} = 0.912$

(e) $r = \log 3 = 0.477 < 1$, $S_\infty = \frac{1}{1-\log 3} = 1.91$