EXERCISES [MAI 1.12-1.13]

MATRICES

SOLUTIONS

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A. Paper 1 questions (SHORT)

MATRIX OPERATIONS

1.

$\begin{pmatrix} 3 & 3-x \\ 2-y & 1 \end{pmatrix}+\begin{pmatrix} 2 & x \\ 1+y & 0 \end{pmatrix}$	$\begin{pmatrix} 5 & 3 \\ 3 & 1 \end{pmatrix}$
$-2\begin{pmatrix} 0 & 1 \\ -4 & 2 \\ 4 & 5 \end{pmatrix}+3\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 0 \end{pmatrix}$	$\begin{pmatrix} 0 & -2 \\ 8 & -4 \\ -8 & -10 \end{pmatrix}+\begin{pmatrix} 3 & 0 \\ 0 & 3 \\ 3 & 0 \end{pmatrix}=\begin{pmatrix} 3 & -2 \\ 8 & -1 \\ -5 & -10 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \end{pmatrix}+\begin{pmatrix} 0 & 0 \end{pmatrix}$	undefined
$\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} 5 & 6 \\ 7 & 8 \end{pmatrix}$	$\begin{pmatrix} 19 & 22 \\ 43 & 50 \end{pmatrix}$
$\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}$	$\begin{pmatrix} 14 \end{pmatrix}$
$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 4 & 6 \\ 3 & 6 & 9 \end{pmatrix}$

2. (a)

$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$
$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	Not possible
$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$	$\begin{pmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{pmatrix}$

(b) $AI_3 = A$, I_3A is not defined, $I_2A = A$

3. (a) (i) $a = 5$

(ii) $b + 9 = 4 \Rightarrow b = -5$

(b) Comparing elements $3(2) - 5(q) = -9 \Rightarrow q = 3$

4. (i) $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} 4 & 2 \\ 1 & 0 \end{pmatrix}$ (ii) $-3\boldsymbol{A} = \begin{pmatrix} -3 & -6 \\ -9 & 3 \end{pmatrix}$ (iii) $\boldsymbol{AB} = \begin{pmatrix} -1 & 2 \\ 11 & -1 \end{pmatrix}$

5. (a) $\boldsymbol{A} + \boldsymbol{B} = \begin{pmatrix} a & b \\ c & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ d & e \end{pmatrix} = \begin{pmatrix} a+1 & b \\ c+d & e \end{pmatrix}$

(b) $\boldsymbol{AB} = \begin{pmatrix} a+bd & be \\ c & 0 \end{pmatrix}$

6. (a) $2x = x + 4$,

$-y = -4 + x + y$

$3z = -1 + z + w$

$w = 2w + 3$

Hence,

$x = 4$, $w = -3$, $y = 0$, $z = -2$

7. (a) $AB = \begin{pmatrix} 7 & 16 & 25 \\ 1 & 13 & 25 \end{pmatrix}$

(b) $AB_1 = \begin{pmatrix} 7 \\ 1 \end{pmatrix}$, $AB_2 = \begin{pmatrix} 16 \\ 13 \end{pmatrix}$, $AB_3 = \begin{pmatrix} 25 \\ 25 \end{pmatrix}$

AB_1, AB_2, AB_3 are the columns of AB

8. The a_{11}-element in AB is $x + 7$ while the a_{11}-element in BA is $x + 8$

So the two matrices cannot be equal.

9. (a) $AB = BA \Leftrightarrow \begin{pmatrix} 1 & 2x \\ 3 & 4x \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 3x & 4x \end{pmatrix} \Leftrightarrow x = 1$

only the identity matrix I,

(b) $AB = BA \Leftrightarrow \begin{pmatrix} x & 2y \\ 3x & 4y \end{pmatrix} = \begin{pmatrix} x & 2x \\ 3y & 4y \end{pmatrix} \Leftrightarrow x = y$

all matrices of the form $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$

10. (a) $A^2 = A$. In general, $A^n = A$ for any $n \in Z^+$

(a) $B^2 = I$. In general, $B^n = I$ if n is even, $B^n = B$ if n is odd.

(b) $C^3 = \boldsymbol{O}$. In general, $C^n = \boldsymbol{O}$ for any $n \geq 3$

11. (a) $\begin{pmatrix} 1 & x & -1 \\ 3 & 1 & 4 \end{pmatrix} \begin{pmatrix} 3 \\ x \\ 2 \end{pmatrix} = \begin{pmatrix} 3 + x^2 - 2 \\ 9 + x + 8 \end{pmatrix} = \begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix}$

(b) $2\begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} = \begin{pmatrix} 20 \\ 28 \end{pmatrix} \Leftrightarrow \begin{pmatrix} 1 + x^2 \\ 17 + x \end{pmatrix} = \begin{pmatrix} 10 \\ 14 \end{pmatrix} \Leftrightarrow x = -3$

12. $\boldsymbol{A}^2 = \begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ k & -1 \end{pmatrix} = \begin{pmatrix} 1+2k & 0 \\ 0 & 2k+1 \end{pmatrix}$

$1 + 2k = 0 \Rightarrow k = -\frac{1}{2}$

13. $\boldsymbol{AB} = \begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix}\begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix} = \begin{pmatrix} 2x+32 & xy+16 \\ 24 & 4y+8 \end{pmatrix}$ $\quad$ $\boldsymbol{BA} = \begin{pmatrix} 2 & y \\ 8 & 4 \end{pmatrix}\begin{pmatrix} x & 4 \\ 4 & 2 \end{pmatrix} = \begin{pmatrix} 2x+4y & 2y+8 \\ 8x+16 & 40 \end{pmatrix}$

$\boldsymbol{AB} = \boldsymbol{BA} \Rightarrow 8x + 16 = 24$ and $4y + 8 = 40$

This gives $x = 1$ and $y = 8$.

14. (a) $\boldsymbol{WP} = \begin{pmatrix} 13 \\ 5 \\ 6 \end{pmatrix}$

(b) $2\boldsymbol{WP} + \boldsymbol{S} = \boldsymbol{Q}$

$\Rightarrow \boldsymbol{S} = \boldsymbol{Q} - 2\boldsymbol{WP} = \begin{pmatrix} 26 \\ 12 \\ 10 \end{pmatrix} - \begin{pmatrix} 26 \\ 10 \\ 12 \end{pmatrix} = \begin{pmatrix} 0 \\ 2 \\ -2 \end{pmatrix}$

DETERMINANT AND INVERSE MATRIX

15.

Matrix	Determinant	Inverse
$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$
$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$	– 1	$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} = A$
$B = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix}$
$C = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$	1	$\begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix}$
$D = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$	– 1	$\begin{pmatrix} 0 & 1 \\ 1 & -1 \end{pmatrix}$
$E = \begin{pmatrix} 0 & 1 \\ 1 & 1 \end{pmatrix}$	– 1	$\begin{pmatrix} -1 & 1 \\ 1 & 0 \end{pmatrix}$
$F = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$	0	There is no inverse

16.

Matrix	Determinant	Inverse
$A = \begin{pmatrix} 2 & 5 \\ -2 & 7 \end{pmatrix}$	24	$A^{-1} = \frac{1}{24}\begin{pmatrix} 7 & -5 \\ 2 & 2 \end{pmatrix}$
$B = \begin{pmatrix} 1 & 5 \\ 2 & 10 \end{pmatrix}$	0	There is no inverse
$C = \begin{pmatrix} 3 & 0 \\ 0 & 5 \end{pmatrix}$	15	$C^{-1} = \begin{pmatrix} \frac{1}{3} & 0 \\ 0 & \frac{1}{5} \end{pmatrix}$

17. $2p^2 + 12p = 14 \Leftrightarrow p^2 + 6p - 7 = 0 \Leftrightarrow (p + 7)(p - 1) = 0$

$p = -7$ or $p = 1$

18. $\begin{vmatrix} k-4 & 3 \\ -2 & k+1 \end{vmatrix} = 0 \Leftrightarrow (k - 4)(k + 1) + 6 = 0 \Leftrightarrow k^2 - 3k + 2 = 0 \Leftrightarrow (k - 2)(k - 1) = 0$

$k = 2$ or $k = 1$

19. (a) $AB = \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix}\begin{pmatrix} a^{-1} & 0 \\ 0 & b^{-1} \end{pmatrix} = \begin{pmatrix} aa^{-1} & 0 \\ 0 & bb^{-1} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$

(b) $M = \begin{pmatrix} 3 & 0 \\ 0 & 1/5 \end{pmatrix} \Rightarrow M^{-1} = \begin{pmatrix} 1/3 & 0 \\ 0 & 5 \end{pmatrix}$

20. (a) $\det A = 1$

(b) $x = 0$

(c) A and B are inverse to each other, so $A^{-1} = B$ and $B^{-1} = A$

21. (a) $2\boldsymbol{A} = \begin{pmatrix} 6 & 4 \\ 2k & 8 \end{pmatrix}$

$2\boldsymbol{A} - \boldsymbol{B} = \begin{pmatrix} 6 & 4 \\ 2k & 8 \end{pmatrix} - \begin{pmatrix} 2 & 2 \\ 1 & 3 \end{pmatrix} = \begin{pmatrix} 4 & 2 \\ 2k-1 & 5 \end{pmatrix}$

(b) det $(2\boldsymbol{A} - \boldsymbol{B}) = 20 - 2(2k - 1) = 22$ □ $4k$

22. (a) $\boldsymbol{CD} = \begin{pmatrix} -2 & 4 \\ 1 & 7 \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix} = \begin{pmatrix} -14 & -4+4a \\ -2 & 2+7a \end{pmatrix}$

(b) $\det \boldsymbol{D} = 5a + 2$, $\boldsymbol{D}^{-1} = \frac{1}{5a+2}\begin{pmatrix} a & -2 \\ 1 & 5 \end{pmatrix}$

(c) $\boldsymbol{D}^2 = \begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix}\begin{pmatrix} 5 & 2 \\ -1 & a \end{pmatrix} = \begin{pmatrix} 23 & 10+2a \\ -5-a & a^2-2 \end{pmatrix}$

23. (a) $\boldsymbol{AB} = \begin{pmatrix} 1 & -2 \\ 3 & p \end{pmatrix}\begin{pmatrix} -2 & 1 \\ q & \frac{1}{2} \end{pmatrix} = \begin{pmatrix} -2-2q & 0 \\ -6+pq & 3+\frac{p}{2} \end{pmatrix}$

(b) **METHOD 1**

using $\boldsymbol{AB} = \boldsymbol{I}$

$-2 - 2q = 1$ and $3 + \frac{p}{2} = 1$, (or $-6 + pq = 0$)

$p = -4,\ q = -\frac{3}{2}$

METHOD 2

finding $\boldsymbol{A}^{-1} = \frac{1}{p+6}\begin{pmatrix} p & 2 \\ -3 & 1 \end{pmatrix}$

$\boldsymbol{A}^{-1} = \boldsymbol{B}$

$\frac{2}{p+6} = 1$ and $-\frac{3}{p+6} = q$

$p = -4,\ q = -\frac{3}{2}$

24. (a) determinant $= 0 \Leftrightarrow -9 + 2x = 0 \Leftrightarrow 2x = 9 \Leftrightarrow x = \frac{9}{2}$

(b) **METHOD 1**

$\boldsymbol{A}^{-1} = \frac{1}{-9+2x}\begin{pmatrix} -3 & -x \\ 2 & 3 \end{pmatrix}$

Since $\boldsymbol{A} = \boldsymbol{A}^{-1}$

$-3 = 3(-9 + 2x) \Leftrightarrow x = 4$

METHOD 2

$\boldsymbol{A}^2 = \boldsymbol{I}$

$\boldsymbol{A}^2 = \begin{pmatrix} 9-2x & 0 \\ 0 & -2x+9 \end{pmatrix}$

$9 - 2x = 1 \Leftrightarrow x = 4$

25. We have to show that $(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2) = \boldsymbol{I}$

$(\boldsymbol{I} - \boldsymbol{X})(\boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2) = \boldsymbol{I}^2 + \boldsymbol{IX} + \boldsymbol{IX}^2 - \boldsymbol{XI} - \boldsymbol{X}^2 - \boldsymbol{X}^3$

$= \boldsymbol{I} + \boldsymbol{X} + \boldsymbol{X}^2 - \boldsymbol{X} - \boldsymbol{X}^2 - \boldsymbol{X}^3$

$= \boldsymbol{I} - \boldsymbol{X}^3 = \boldsymbol{I}$

26. (a) det $\boldsymbol{A} = -2$

(b) $\boldsymbol{A}^{-1} =$

(a) det $\boldsymbol{A}^n = (-2)^n$

B. Paper 2 questions (*LONG*)

27.

	Wrong answer	Correct answer
$\boldsymbol{AX} + \boldsymbol{BX} =$	$\boldsymbol{X}(\boldsymbol{A}+\boldsymbol{B})$	$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}$
$\boldsymbol{XA} + \boldsymbol{XB} =$	$(\boldsymbol{A}+\boldsymbol{B})\boldsymbol{X}$	$\boldsymbol{X}(\boldsymbol{A}+\boldsymbol{B})$
$\boldsymbol{AB} + \boldsymbol{A} =$	$\boldsymbol{A}(\boldsymbol{B}+1)$	$\boldsymbol{A}(\boldsymbol{B}+\boldsymbol{I})$
$\boldsymbol{AB} + 2\boldsymbol{A} =$	$(\boldsymbol{B}+2\boldsymbol{I})\boldsymbol{A}$	$\boldsymbol{A}(\boldsymbol{B}+2\boldsymbol{I})$
$\boldsymbol{BA} + k\boldsymbol{A} =$	$(\boldsymbol{B}+k)\boldsymbol{A}$	$(\boldsymbol{B}+k\boldsymbol{I})\boldsymbol{A}$
$\boldsymbol{A}^2 + \boldsymbol{A} =$	$\boldsymbol{A}(\boldsymbol{A}+1)$	$\boldsymbol{A}(\boldsymbol{A}+\boldsymbol{I})$
$(\boldsymbol{B}+\boldsymbol{C})\boldsymbol{A} =$	$\boldsymbol{AB} + \boldsymbol{AC}$	$\boldsymbol{BA} + \boldsymbol{CA}$
$(\boldsymbol{A}+\boldsymbol{C})\boldsymbol{A} =$	$\boldsymbol{A}^2 + \boldsymbol{AC}$	$\boldsymbol{A}^2 + \boldsymbol{CA}$
$(\boldsymbol{A}+\boldsymbol{B})^2 =$	$\boldsymbol{A}^2 + 2\boldsymbol{AB} + \boldsymbol{B}^2$	$\boldsymbol{A}^2 + \boldsymbol{AB} + \boldsymbol{BA} + \boldsymbol{B}^2$
$(\boldsymbol{A}+\boldsymbol{I})^2 =$	$\boldsymbol{A}^2 + 2\boldsymbol{A} + 1$	$\boldsymbol{A}^2 + 2\boldsymbol{A} + \boldsymbol{I}$

28. (a) $A^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$, $A^3 = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$, $A^4 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$

(b) $A^n = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}$

(c) $A^{10} = \begin{pmatrix} 1 & 20 \\ 0 & 1 \end{pmatrix}$ (the guess in (b) is true for n = 10)

(d) $A^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = I$. It gives the identity matrix.

(e) $A^{-1} = \begin{pmatrix} 1 & -2 \\ 0 & 1 \end{pmatrix}$, it satisfies the general formula of A^n for $n = -1$

(f) $A^{-n} = \begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix}$

$$A^n A^{-n} = \begin{pmatrix} 1 & 2n \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 1 & -2n \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \boldsymbol{I}$$

so A^{-n} is the inverse of A^n.

29. (a) (i) $S_4 = 10$

(ii) $S_{100} = 10100$

(b) (i) $\boldsymbol{M}^2 = \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix}$

(ii) $\boldsymbol{M}^3 = \boldsymbol{M}^2\,\boldsymbol{M} = \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix}$

(c) (i) $\boldsymbol{M}^4 = \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix}$

(ii) $\boldsymbol{T}^4 = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 6 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 8 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 4 & 20 \\ 0 & 4 \end{pmatrix}$

(d) $\boldsymbol{T}_{100} = \begin{pmatrix} 1 & 2 \\ 0 & 1 \end{pmatrix} + \begin{pmatrix} 1 & 4 \\ 0 & 1 \end{pmatrix} + \ldots + \begin{pmatrix} 1 & 200 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} 100 & 10100 \\ 0 & 100 \end{pmatrix}$